Contents

Chapter 6

Holographic Renormalization

6.1 Holographic Regularization

In the previous chapter we have seen in detail how Lorentzian correlation functions are defined in field theory and how the general framework is represented holographically. We will make use of this Lorentzian framework in the next chapter when we determine transport properties of theories with holographic duals. Before we turn our attention to these applications we shall spend some time to formalize the framework of holographic renormalization[1] that we alluded to briefly further above. For simplicity we will work in Euclidean signature, but the formalism carries over to the Lorenztian signatures, mutatis mutandis (see previous chapter). Our starting point is the relationship between the on-shell gravitational action and the field-theory generating functional of connected correlators,

$$W_{\rm QFT}[\phi_{(0)}] = -S^E_{\text{on-shell}}[\Phi \to \phi_{(0)}]\,. \tag{6.1}$$

We have already seen that we encounter UV divergences in evaluating this expression. By cutting off the integrals in the UV, that is near the AdS boundary, in general, these look like

$$S_{\rm reg}[f_{(0)};t] = \int_\epsilon d^dx\sqrt{g_{(0)}}\left[\epsilon^{-\nu}a_{(0)} + \epsilon^{-(\nu-1)}a_{(2)} + \cdots - \log(\epsilon)a_{(2\nu)} + \text{finite}\right]\ , \tag{6.2}$$

where 'reg' stands for regularized, as we should essentially think of the above expression as a regulated quantity in the field theory sense. We already saw how

[1] Our treatment here is based closely on that of [1]. These lecture notes contain more related material and references to the original literature with two classic references being [2] and [3].

removing the regulator results in divergent contributions to the free energy, and we also introduced an ad-hoc way to deal with this by noting that physically we are only interested in free-energy differences. However, the divergences also affect $n-$point functions, where it is not obvious that such a background subtraction scheme is particularly well motivated. It will pay to put things on a more solid footing. We will study:

1. How to define a regularized on-shell action $S_{\mathrm{reg}}[\cdots;\epsilon]$
2. How to add counterterms $S_{\mathrm{CT}}[\cdots;\epsilon] = -\mathrm{div}(S_{\mathrm{reg}}[\cdots;\epsilon])$ to the regulated on-shell action
3. How to extract renormalized $n-$point functions from the above which are cutoff independent
4. How to understand RG transformation of $n-$point functions holographically

In order to understand the first item we need to define what we mean by a spacetime that asymptotically looks like anti-de Sitter space. Such a spacetime is also often called asymptotically locally AdS. By extracting the universal behavior of such spaces near the boundary we will be able to define a UV regulator, as required. In order to do so we will introduce some notions from differential geometry of asymptotically anti-de Sitter spaces.

6.1.1 Asymptotic AdS and Fefferman-Graham expansion

The manifold $\mathcal{M}$ =AdS$_{d+1}$ is maximally symmetric, which means that, in a local chart, we can write the Riemann tensor in terms of the metric as

$$R_{abcd} = \frac{1}{\ell^2}\left(G_{ac}G_{bd} - G_{ad}G_{ba}\right). \tag{6.3}$$

It is useful to rewrite the write the global metric (see Eq. (**??**)) as

$$ds^2 = \frac{\ell^2}{\cos^2\theta}\left(-dt^2 + d\theta^2 + \sin^2\theta d\Omega^2_{d-1}\right), \tag{6.4}$$

with $\tan\theta = \sinh\rho$, where $0 \le \theta < \frac{\pi}{2}$ and ρ is the 'normal' radius of global AdS$_{d+1}$. We note that (6.4) has a second-order pole at $\theta = \frac{\pi}{2}$, which corresponds in the present coordinate system to UV boundary. However, this observation allows us to strip-off the boundary metric, by compensating this pole with a conformal factor

$$g_{(0)} := r^2 G\big|_{\theta=\pi/2}\,. \tag{6.5}$$

Strictly speaking, however, this does not define an induced metric on $\partial\mathcal{M}$, but instead a conformal class of boundary metrics, corresponding to the class of functions $r(x^\mu)$ which have a simple zero at the UV boundary. We define

$$r(x) = \begin{cases} r(x) > 0 \text{ in interior} \\ r \text{ has simple zero at } \pi/2 \end{cases}$$

One example would be $r = \cos\theta$. However, if r satisfies the requisite properties to be a defining function, then so does re^ω where ω has no zeros or poles at $\pi/2$. Therefore,

$$[g_{(0)}] = \ell^2 \eta_{\mu\nu} dx^\mu dx^\nu \cong \ell^2 e^\omega \eta_{\mu\nu} dx^\mu dx^\nu \tag{6.6}$$

are equally valid boundary metrics, where one is related to the other via a conformal rescaling[2]. This is the precise reason why AdS_{d+1} admits a conformal class of boundary metrics, rather than simply *a particular* boundary metric. Everything that was said up to now referred to the case of anti-de Sitter space itself. We shall now extend the discussion to spaces which merely approach the geometry of AdS when we look at the space near the boundary.

We will take the behavior established above as the model of our definition of an aAdS ('conformally compact' manifold or 'asymptotically locally AdS' manifold). Thus, let us define the conformal compactification of a manifold X as the process of equipping the asymptotic boundary with a conformal class of metrics, defining in some sense the closure $\overline{X}$ of the 'interior' X. Thus we define the operation of conformal compactification as

$$\underset{\text{interior}}{X} \longrightarrow \underset{\text{conf. compact.}}{\overline{X}} \tag{6.7}$$

such that the conformal metric $g = r^2 G$ extends smoothly to $\bar{X} \cup \partial X$. By smoothly we mean, of course, that the double pole in the 'bare' metric G is compensated by the double zero in r^2. If we then demand that the metric satisfy the Einstein equations,

$$R_{ab} - \frac{1}{2} R g_{ab} = \Lambda G_{ab}\,, \tag{6.8}$$

we find that the Riemann tensor behaves asymptotically near the UV boundary as

$$R_{abcd} = \frac{1}{\ell^2}\left(G_{ac}G_{bd} - G_{ad}G_{bc}\right) + O(r^{-3}). \tag{6.9}$$

[2]Whether or not the boundary field theory is invariant under this conformal scaling is an interesting question. If it is not, the theory is said to have a conformal anomaly. See [1] for more details.

We can choose the function r as the radial coordinate, writing the metric in a near-boundary expansion as

$$ds^2 = \frac{\ell^2}{r^2}\left(dr^2 + g_{\mu\nu}(x,r)dx^\mu dx^\nu\right) . \qquad (6.10)$$

For the remainder of this chapter it will be convenient to adopt units with $\ell = 1$. The function $g_{\mu\nu}(x,r)$ itself has an expansion of the form

$$g_{\mu\nu}(x,r) = g_{(0)\mu\nu} + r g_{(1)\mu\nu} + r^2 g_{(2)\mu\nu} + \cdots , \qquad (6.11)$$

where $g_{(A)\mu\nu}$ are determined order by order from (6.8). In the literature often a coordinate $\rho = r^2$ is used, in terms of which we have

$$ds^2 = \frac{d\rho^2}{4\rho^2} + \frac{1}{\rho} g_{\mu\nu}(x,\rho)dx^\mu dx^\nu \qquad (6.12)$$

with

$$g(x,\rho) = g_{(0)} + \cdots + \rho^{d/2} g_{(d)} + h_{(d)}\rho^{d/2}\log\rho\, . \qquad (6.13)$$

This is known as the Fefferman-Graham expansion after the two mathematicians who first developed it.

Comments

1. We have thus far only considered the vacuum Einstein equations (with cosmological constant), but one can add matter to this process so long as the corresponding matter stress tensor T_{ab} contributes at leading or sub-leading order in the ρ expansion. We will soon see that this corresponds to matter fields which are dual to marginal or relevant operators, respectively, in the boundary field theory. If we added an irrelevant operator, the contribution of the corresponding matter field to the stress energy tensor would be dominant as compared to the cosmological constant. This would render the whole procedure inconsistent and we would have to replace the asymptotically AdS type behavior with a different asymptotic geometry. This is, of course, consistent with expectations from field theory, where irrelevant operators destroy the UV behavior.

2. The point of this differential geometric exercise was to establish a universal notion of the UV behavior of asymptotically AdS geometries. In particular all such manifolds look like AdS near the boundary with small deviations characterized by the higher coefficients $g_{(A)\mu\nu}$. In field theory terms this is the universal behavior of UV fixed points and small deformations about them

(marginal, relevant, irrelevant).

3. The present construction also gives us a well-defined notion of UV regulator: Instead of integrating expressions all the way out to the boundary, let us cut off integrals at a radius $\rho = \epsilon$. We should think of this as part of an RG scheme, and we have shown in detail how this part of our scheme is defined for aAdS spaces. This closely parallels the cut off procedure in ordinary RG, such as a momentum cut off or Pauli-Villars or dimensional regularization. Of course the rest of the scheme still needs to be specified.

Let us now specify in the abstract the renormalization scheme most commonly used in the holographic literature. We will then illustrate the procedure with a concrete example, namely a scalar operator dual to a real scalar field in AdS.

6.2 Holographic Renormalization: Step by Step

Since the generating functional involves the gravity partition function with sources, we need to understand the Dirichlet problem in AdS. The Dirichlet data, i.e. the asymptotic behavior of the fields at the boundary, should be thought of as the sources in the field theory. We have already set up the appropriate formalism that tells us how to thing about this: In order to give Dirichlet boundary conditions for the metric g, we should specify $g_{(0)\mu\nu}(x^\mu)$, as an arbitrary function of the boundary coordinates.

The bulk field content, mirroring the operator content of the dual field theory, contains modes with near boundary expansions of the form

$$\mathcal{F}(x,\rho) = \rho^m \left(f_{(0)}(x) + f_{(2)}(x)\rho + \cdots + \rho^n [f_{(2n)}(x) + \log\rho \tilde{f}_{(2n)}(x) + \cdots] \right) . \tag{6.14}$$

These operators could be bosonic or fermionic and carry various interesting representations of the Lorentz group (scalar, spinor, vector, tensor, ...). We have already encountered an example in Eq. (6.11) above for the metric. The exponents m and n are determined via an asymptotic analysis of the bulk equations of motion. Let us proceed with the generic form and explain the meaning of the individual terms.

Firstly, the leading behavior, $f_{(0)}$, is the Dirichlet data piece and acts as a 'source' for the operator $\mathcal{O}_\mathcal{F}$ in the dual field theory. For the case of the metric in Eq. (6.11) the data $g_{(0)\mu\nu}$ acts as a source for the energy momentum tensor T_{ij} in the boundary field theory.

The sub-leading pieces $f_{(w)}, \ldots, f_{(2n-2)}$ are determined by the bulk equations of motion. These also determine the coefficient $\tilde{f}_{2n}$, which is related to the conformal anomaly. A theory with non-vanishing $\tilde{f}_{2n}$ is not invariant under a change of representative of the conformal class of the boundary metric. In other words, if we perform a conformal transformation of the boundary metric, correlation functions or free energies pick up an anomalous piece. The information contained in these expansion coefficients is enough to determine the divergent coefficients $a_{(A)}$ in (6.2) in terms of $f_{(0)}$, that is we have $a_{(A)}[f_{(0)}]$. This allows us to write down the regularized action, isolating the pieces which diverge when the cutoff is removed.

We then proceed to the next step in our renormalization procedure, the addition of counterterms. These are designed so as to cancel exactly the divergent pieces in the regularized action. One therefore has the subtracted action

$$S_{\text{sub}} = S_{\text{reg}}[f_{(0)}, \epsilon] + S_{\text{ct}}[f_{(0)}, \epsilon], \tag{6.15}$$

where

$$S_{\text{ct}}[f_{(0)}, \epsilon] = -\text{div}\left(S_{\text{reg}}[f_{(0)}, \epsilon]\right) \,. \tag{6.16}$$

It may be a useful analogy to compare this to the 'minimal subtraction' scheme in dimensional regularization of quantum field theories. In fact, I will refer to this procedure as 'holographic minimal subtraction'. The resulting object is finite when the cutoff is removed and defines the generating functional of renormalized correlation functions

$$S_{\text{ren}} = \lim_{\epsilon \to 0} S_{\text{sub.}}[f_{(0)}, \epsilon] = -W^{\text{ren.}}_{\text{QFT}}[f_{(0)}] \,. \tag{6.17}$$

Differentiating with respect to sources generates as usual these correlation functions. For example

$$\langle \mathcal{O}_{\mathcal{F}} \rangle = \frac{1}{\sqrt{g_{(0)}}} \frac{\delta S_{\text{ren}}}{\delta f_{(0)}} \sim f_{(2n)}. \tag{6.18}$$

The exact coefficient depends on normalization choices and such details and we will determine it for the simplest case of a scalar operator in the next section.

Comment

The coefficient $f_{(2n)}$, which is dual to the expectation value of the operator $\mathcal{O}_{\mathcal{F}}$ is not determined by near boundary analysis. In order to determine it we need full solution and in particular an infra-red boundary condition which together with the Dirichlet condition in the UV uniquely determines the solution. Let us now see how this somewhat abstract discussion plays out in the simplest example, namely a spinless bosonic operator $\mathcal{O}_\Phi$ dual to a bulk scalar field $\Phi(\rho, x)$.

6.2.1 Example: Scalar Operator

A spinless bosonic operator is dual to a real scalar field in the bulk. We have the dual relationships

$$\text{scalar field } \Phi \leftrightsquigarrow \text{spinless operator } \mathcal{O}_\Phi$$

$$\text{mass } m \leftrightsquigarrow \text{conformal dim. } \Delta_\Phi$$

The bulk scalar field obeys equations of motion following from the action[3]

$$S = \frac{1}{2}\int d^{d+1}x\sqrt{G}(G^{ab}\partial_a\Phi\partial_b\Phi + m^2\Phi^2)\,. \tag{6.19}$$

Explicitly, they take the form of a wave equation, viz.

$$(-\Box_G + m^2)\Phi = -\frac{1}{\sqrt{G}}\partial_a(\sqrt{G}G^{ab}\partial_b\Phi) + m^2\Phi = 0\,. \tag{6.20}$$

In fact we should really solve the coupled system $S_{\text{grav}} + S[\Phi]$, i.e. the Einstein equations coupled to a scalar field. For simplicity here we use the so-called probe approximation, that is we neglect the backreaction of the scalar field on the geometry. In fact, on the level of the two-point function, this gives us the full result. So let us find a solution that behaves asymptotically as $\Phi = \rho^a\phi(x,\rho)$. With foresight, let us parametrize this exponent as $a = \frac{1}{2}(d-\Delta)$. The field $\phi(x,\rho)$ has a regular expansion in power of ρ,

$$\phi(x,\rho) = \phi_{(0)} + \rho\phi_{(2)} + \rho^2\phi_{(4)} + \cdots\,. \tag{6.21}$$

As usual the exponent a, or equivalently the constant Δ are determined by the indicial equation, which in the present case reads

$$m^2 - \Delta(\Delta - d) = 0 \tag{6.22}$$

In fact, we chose the notation in terms of Δ since this equation determines conformal dimension of the operator $\mathcal{O}_\Phi$ in terms of the bulk mass m. We shall return to this interpretation below. Having stripped off this asymptotic behavior, the

[3] Recall that we are using Euclidean signature in this chapter. As we have seen several times now, the main subtlety lies in the infra-red boundary condition, and we avoid this by sticking to the Euclidean case. The near-boundary analysis carries over unchanged to Lorentzian signature and so in particular the divergences are treated in exactly the same way.

remaining equation for $\phi(x,\rho)$ is

$$0 = \underbrace{\delta^{ij}\partial_i\partial_j\phi}_{\Box_0\phi} + 2(d-2\Delta+2)\partial_\rho\phi + 4\,\rho\,\partial_\rho^2\phi\,. \tag{6.23}$$

This equation can now be solved straightforwardly, order by order in the ρ expansion. At leading order we find that the coefficient $\phi_{(2)}$ is fully determined in terms of the source,

$$\phi_{(2)} = \frac{1}{2(2\Delta-d-2)}\Box_0\phi_{(0)}\,. \tag{6.24}$$

Carrying out this process to higher order results in the expressions

$$\begin{aligned} \phi_{(2)} &= \frac{1}{2(2\Delta-d-2)}\Box_0\phi_{(0)} \\ \phi_{(4)} &= \frac{1}{4(2\Delta-d-4)}\Box_0\phi_{(2)} \\ &\vdots \\ \phi_{(2n)} &= \frac{1}{2n(2\Delta-d-2n)}\Box_0\phi_{(2n-2)} \end{aligned} \tag{6.25}$$

where, crucially, all these coefficients are determined algebraically in terms of the source. In other words, once the Dirichlet data is specified, all the above coefficients are uniquely determined. Attempting to push the expansion further, we need to distinguish between two cases.

I) So long as $2\Delta-d$ is not an integer, the second solution with leading power $\rho^{\Delta/2}\phi_{2\Delta-d}$ is not determined in terms of the source $\phi_{(0)}$. Another way of saying the same thing is that the two asymptotic solution branches are linearly independent. As we shall see, the coefficient $\phi_{2\Delta-d}$ plays the role of he expectation value of the dual operator. As before this should not be determined by the asymptotic analysis alone, but instead we need infra-red input.

II) If, however, $2\Delta-d$ is integer, the solutions with power $\rho^{\Delta/2}$ and $\rho^{\frac{1}{2}(d-\Delta)}$ are linearly dependent. Thus, in order to have two linearly independent solutions – as is required for a second-order equation – it is necessary to introduce a logarithmic term at order $O(\rho^{\Delta/2})$:

$$\rho^{\Delta/2}\left(\phi_{(2\Delta-d)} + \psi_{(2\Delta-d)}\log\rho + \cdots\right), \tag{6.26}$$

with

$$\psi_{(2\Delta-d)} = -\frac{1}{2^{2k}\Gamma(k)\Gamma(k+1)}(\Box_0)^k\phi_{(0)}, . \tag{6.27}$$

Again the expectation value $\phi_{(2\Delta-d)}$ is not determined by $\phi_{(0)}$. The logarithmic coefficient $\psi_{(2\Delta-d)}$ contributes to the conformal anomaly $\mathcal{A}$ of the dual field theory.

We now have determined the solutions in an expansion near the UV boundary, leaving undetermined one of the two linearly independent solution branches. This structure contains all the necessary information in order to determine the regularized action, and in particular the divergent part. We evaluate the on-shell action in terms of our asymptotic series, arriving at the expression

$$\begin{aligned} S_{\text{reg}} &= \frac{1}{2}\int_{\rho\geq\epsilon} d^{d+1}x\sqrt{G}\left((\partial\Phi)^2 + m^2\Phi^2\right) \\ &= \int_{\rho=\epsilon} d^dx\left(\epsilon^{-\Delta+\frac{d}{2}}a_{(0)} + \epsilon^{(-\Delta+\frac{d}{2}+1)}a_{(2)} + \cdots\right), \end{aligned}$$

where the coefficients of the divergent pieces are determined by the asymptotic expansion as local functions of the source

$$a_{(0)} = -\frac{1}{2}(d-\Delta)\phi_{(0)}^2; \qquad a_{(2)} = -\frac{d-\Delta+1}{2(2\Delta-d-2)}\phi_{(0)}\Box_0\phi_{(0)}.$$

Next on our to-do list is to find the corresponding local counterterms that cancel these divergences. In order to achieve this, it is easiest to just invert the asymptotic expansion to a given order. We first state the resulting counterterms and carry out the calculation later. The statement is then that all divergences present in the regularized action are cancelled by the counterterms

$$S_{\text{ct}} = \int\sqrt{\gamma}\left(\frac{d-\Delta}{2}\Phi^2 + \frac{1}{2(2\Delta-d-2)}\Phi\Box_\gamma\Phi\right). \tag{6.28}$$

Then we define the renormalized

$$S_{\text{ren}} = \lim_{\epsilon\to 0}\left(S_{\text{sub}} - S_{\text{ct}}\right) \tag{6.29}$$

which has, by construction, a finite limit as the cutoff is removed. A calculation shows that

$$\frac{\delta S_{\text{ren}}}{\delta \phi_{(0)}} := \langle O_\Phi \rangle = -(2\Delta - d)\phi_{(2\Delta-d)} + C(\phi_{(0)})\,, \tag{6.30}$$

where the last term is the scheme-dependent part, which can be removed with finite counter-terms. The expectation value, as presaged several times above is determined by the coefficient $\phi_{(2\Delta-d)}$, which itself is *not* determined by the Dirichlet data $\phi_{(0)}$ at the boundary. It will be determined from matching the UV behavior to the appropriate IR solution in order to construct the full bulk field. We now go back to the details of the counterterm action, supplying the calculational details we skipped above. We want to invert

$$\Phi = \rho^{\frac{1}{2}(d-\Delta)} \left(\phi_{(0)} + \frac{\rho}{2(2\Delta - d - 2)} \Box_0 \phi_0 + \cdots \right) \tag{6.31}$$

at $\rho = \epsilon$. This is straightforward and we find

$$\Phi_\epsilon = \epsilon^{\frac{1}{2}(d-\Delta)} \left(\phi_{(0)} + \frac{\epsilon}{2(2\Delta - d - 2)} \Box_0 \phi_0 + \cdots \right). \tag{6.32}$$

Therefore

$$\begin{aligned} \phi_{(0)} &= \epsilon^{\frac{\Delta}{2} - \frac{d}{2}} \Phi_\epsilon - \frac{\epsilon}{2(2\Delta - d - 2)} \Box_0 \phi_{(0)} + \cdots \\ &= \left(\Phi_\epsilon - \frac{1}{2(2\Delta - d - 2)} \underbrace{\epsilon \Box_0}_{\Box_\gamma} \Phi_\epsilon + \cdots \right), \end{aligned}$$

where $\epsilon\,\Box_{(0)} = \epsilon\,\delta^{ij}\partial_i\partial_j$ is the Laplacian[4] with respect to the metric $\gamma_{ij} = \frac{1}{\epsilon}\delta_{ij}$ induced on the cutoff surface. Similarly for $\phi_{(2)}$ one has

[4] If the signature is Lorentzian it would be the d'Alambertian

$$\begin{aligned}\phi_{(2)} &= \frac{1}{2(2\Delta - d - 2)}\Box_0\phi_{(0)} \\ &= \frac{\epsilon^{\frac{\Delta}{2}} - \frac{d}{2}}{2(2\Delta - d - 2)}\Box_0\Phi_\epsilon + O(\Box^2\Phi_\epsilon) \\ &= \frac{\epsilon^{\frac{\Delta}{2}} - \frac{d}{2} - 1}{2(2\Delta - d - 2)}\Box_\gamma\Phi_\epsilon + \cdots .\end{aligned}$$

Substituting these expressions into the regularized action we find that the divergent pieces are exactly (minus) the counterterms shown in Eq. (6.28) above. This proves that the subtracted action indeed is finite upon removal of the cutoff, as required.

6.2.2 Callan-Symanzik Equation

The renormalized action (6.29) defines for us a generating functional for renormalized correlation functions. As usual it is illuminating to study the behavior of the renormalized quantities under a scale transformation. Such a scale transformation acts holographically as

$$\rho \longrightarrow \mu^2\rho : \qquad \text{energy scale} \tag{6.33}$$

$$x^i \longrightarrow \mu x^i : \qquad \text{scale transformation}\,, \tag{6.34}$$

where the power of μ has been fixed by dimensional analysis following our UV analysis. The idea is to find the behavior of the various coefficients in the UV expansion – keeping in mind their interpretation as sources and expectation values in the dual field theory – and then finding the differential equations satisfied by these objects as a function of scale. Such equations are referred to in the literature as Callan-Symanzik equations, and we find here their holographic incarnations. Let us proceed by considering the simplest case, namely that of a spinless bosonic operator, dual to a scalar field in the bulk. By definition, the scalar field $\Phi(x,\rho)$ is invariant under these transformation, i.e.

$$\Phi'(x',\rho') = \Phi(x,\rho)\,. \tag{6.35}$$

By studying the action of the transformation on each term in the expansion (6.21), and imposing the transformation law (6.35), one finds the induced transformations of the coefficient functions. We display the two most interesting ones. The source transforms as

$$\phi'_{(0)}(x') = \mu^{d-\Delta}\phi_{(0)}(x'\mu) \tag{6.36}$$

while the expectation value transforms as

$$\phi'_{(2\Delta-d)}(x') = \mu^{\Delta}(\phi_{(2\Delta-d)}(x'\mu) + \underset{\substack{\text{if log term}\\ \text{present}}}{\log} \mu^2\psi_{(2\Delta-d)}(x'))\,, \tag{6.37}$$

where, as indicated, the inhomogeneous part of the transformation only occurs in cases where there is a log term in the Fefferman-Graham expansion. By taking derivatives with respect to the scale we deduce from (6.36)

$$\mu\frac{\partial}{\partial\mu}\phi_{(0)}(x'\mu) = -(d-\Delta)\phi_{(0)}(x'\mu), \tag{6.38}$$

which expresses the fact that $\mathcal{O}_\Phi$ has dimension Δ. In fact, this can be seen even more clearly by finding the RG equation satisfied by the expectation value. Taking derivatives with respect to scale we have

$$\langle\mathcal{O}(x')\rangle = \mu^{\Delta}\left[\langle\mathcal{O}(x'\mu) - (2\Delta-d)\log\mu^2\psi_{(2\Delta-d)}(x'\mu)\right]. \tag{6.39}$$

That is, the one-point function scales as μ^Δ *up to* possible logarithmic corrections due to *anomaly*. Disregarding this anomalous piece we have precisely the transformation law for the one-point function of an operator of dimension Δ. It should be evident by now that higher n-point functions will satisfy exactly the right RG equations to qualify as n-point functions of operators of dimension Δ, by a procedure analogous to the above.

6.3 Witten Diagrams

In general the computations we have performed are sensitive to the interaction terms that appear in the bulk action representing the dual field theory. So far, for simplicity, we have only really considered the case of a free massive scalar, but as we already commented, for a full treatment we should also take into account its interaction with the gravitational sector. Naturally any (set of) bulk field may also have its self-interactions or interaction vertices with other bulk fields. To convince oneself of this fact, I encourage the reader to briefly consider the case of a non-abelian Yang-Mills field in the bulk which obviously has gluon-gluon vertices in the bulk. We will now outline how bulk interactions are featured in the formalism above. In fact, this is not a conceptual complication, but can lead to a considerable calculational complication. Let us again illustrate what is going on via the simple exmaple of bulk scalar interactions. The algebraic expressions we encounter are liable to become rather unwieldy, so at an appropriate point we will introduce a diagrammatic representation analogous to Feynman diagrams which helps to easier visualize the essential physics.

6.3.1 Bulk Scalar With Self Interaction

For this purpose we generalize our action (6.19) to

$$S = \int d^{d+1}x \left(\sum_{i=1}^{3} \tfrac{1}{2} \left(\partial \Phi_i\right)^2 + \tfrac{1}{2} m^2 \Phi_i^2 + \tfrac{1}{3} \lambda_{ijk} 3 \Phi_i \Phi_j \Phi_k \right) , \tag{6.40}$$

that is we consider a multiplet of three scalar operators $\mathcal{O}_{\Phi_i}$, all with the same conformal dimension Δ which interact via a bulk three-point coupling. The idea is now to solve the equations of motion

$$\left(-\Box + m^2\right) \Phi_i + \lambda_{ijk} \Phi_j \Phi_k = 0 \tag{6.41}$$

perturbatively in $\lambda_{ijk} \sim \mathcal{O}(\lambda)$ with $\lambda \ll 1$. That is we seek

$$\Phi_i = \Phi_i^{(0)} + \Phi_i^{(1)} + \mathcal{O}(\lambda^2) \tag{6.42}$$

where the fields have prescribed boundary conditions

$$\Phi_i \sim \rho^{\frac{d-\Delta}{2}} \phi_i^{(0)} + \cdots . \tag{6.43}$$

We have the equations

$$\begin{aligned} \left(-\Box + m^2\right) \Phi_i^{(0)} &= 0 \, , \\ \left(-\Box + m^2\right) \Phi_i^{(1)} &= 2 \lambda_{ijk} \Phi_j^{(0)} \Phi_k^{(0)} \, . \end{aligned} \tag{6.44}$$

In order to solve them, we introduce two formal objects. Firstly we introduce the bulk to boundary propagator, $K_\Delta^{ij}(\rho, x^\mu - y^\mu) = \delta^{ij} K_\Delta(\rho, x^\mu - y^\mu)$, satisfying

$$\left(-\Box + m^2\right) K_\Delta(\rho, x^\mu - y^\nu) = \rho^{\frac{d-\Delta}{2}} \delta^d(x^\mu - y^\mu) \, , \tag{6.45}$$

and secondly the bulk to bulk propagator

$$\left(-\Box + m^2\right) G_\Delta(x^a - y^a) = \delta^{d+1}(x^a - y^a) \, . \tag{6.46}$$

The former allows us to write the full bulk solution at order λ^0 as a function of the boundary value

$$\Phi_i^{(0)}(\rho, x^\mu) = \int d^d y K_\Delta \left(\rho, x^\mu - y^\mu\right) \phi_i^{(0)}(y^\mu) \, , \tag{6.47}$$

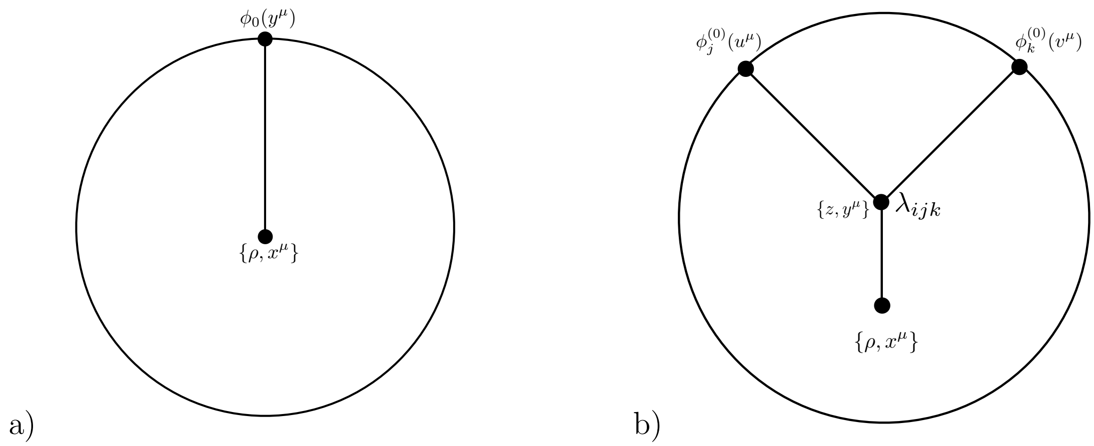


Figure 6.1: Two (incomplete) Witten diagrams showing the two basic contributions ((6.47) in panel a) and ((6.3.1) in panel b) to the bulk solution $\Phi_i(\rho, x^\mu)$. These form the basic building blocks for the so-called Witten diagrams used in the calculation of higher-point functions in holography. Note that all source insertions as well as bulk vertices are integrated over.

while the latter allows us to write the full bulk solution at first order in λ as

$$\Phi_i^{(1)}(\rho, x^\mu) = \lambda_{ijk} \int d^{d+1}y d^d u d^d v G_\Delta(x^a - y^a) K_\Delta(z, y^\mu - u^\mu) \phi_j^{(0)}(u^\mu) K_\Delta(z, y^\mu - u^\mu) \phi_k^{(0)}(v^\mu) \,. \tag{6.48}$$

The first equation is very familiar from our previous two-point function calculations. There we calculated an explicit representation of the bulk-to boundary propagator in momentum space in terms of the mode functions f_k. Here we do not enter into the specifics and instead work with the abstract object K_Δ itself. The second equation expresses the field at an arbitrary bulk point $x^a = (\rho, x^\mu)$ in terms of an integral over the two sources at the boundary. One is inserted at the boundary point u^μ and then propagated into the bulk point $y^a = (z, y^\mu)$ with $K_\Delta(z, y^\mu - u^\mu)$, the second is inserted at the boundary at the point v^μ and then propagated to the same bulk point $y^a = (z, y^\mu)$ with $K_\Delta(z, y^\mu - v^\mu)$. The final ingredient is a further propagation in the bulk via the bulk interaction λ_{ijk} from the point $y^a = (z, y^\mu)$ to the point $x^a = (\rho, x^\mu)$ using the bulk to bulk propagator $G_\Delta(x^a - y^a)$. It is probably best to think about these equations visually, via so called Witten diagrams, which are a holographic analog of Feynman diagrams in ordinary field theory. The two contributions in Eq. (6.47) and (6.3.1) are shown in Figure 6.1.

To go from here to the correlation functions only requires a bit little more work. The bulk solution (6.47)(6.3.1) has the asymptotic behavior of a field of mass m in anti-de Sitter space, so from our holographic RG analysis we know that the renormalized one-point function is given as the expansion coefficient $\phi_{2\Delta-d}$, up to contact terms, the precise relation being given by Eq. (6.30). This means that we extract the one-point function in the presence of sources simply by reading off the appropriate coefficient in the near-boundary expansion of the solution (6.47), (6.3.1) . But once we have the renormalized one-point function in the presence of sources we can generate all higher point functions by further differentiation with respect to sources. In the present context, for example, the four-point function will be given by

$$\langle \mathcal{O}_i(x_1^\mu)\mathcal{O}_j(x_1^\mu)\mathcal{O}_k(x_1^\mu)\mathcal{O}_l(x_1^\mu)\rangle = \left.\frac{\delta^3\langle \mathcal{O}_i\rangle_{\rm ren}}{\delta\phi_j^{(0)}\delta\phi_k^{(0)}\delta\phi_l^{(0)}}\right|_{\phi_i^{(0)}=0} . \tag{6.49}$$

Rather than carrying out this tedious computation explicitly, let us display some contributing Witten diagrams in Fig. 6.2. As a final comment, let us remark that, in principle, this procedure can be pushed beyond tree level, where we treat the bulk gravity solution as an effective field theory (EFT). This way we may compute loop corrections to any n-point function, as illustrated in Fig. Corrections that involve graviton vertices are suppressed by appropriate factors of the bulk Newton constant $G_N \sim 1/N^2$. It should by now be apparent that this procedure produces for us any Euclidean n-point function.

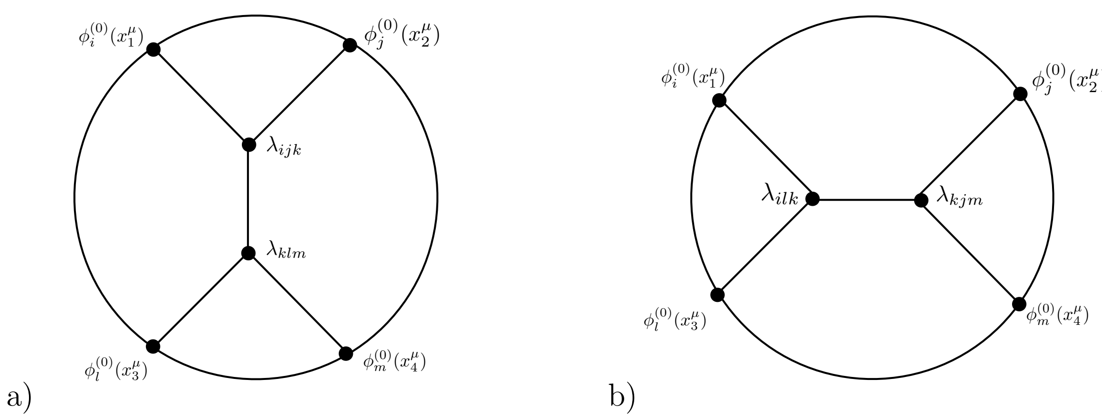
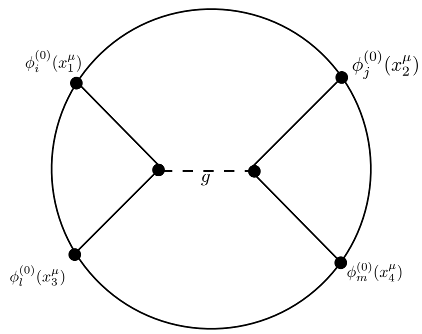


Figure 6.2: Two tree-level Witten diagrams contributing to the boundary four-point function $\langle \mathcal{O}_i\mathcal{O}_j\mathcal{O}_l\mathcal{O}_m\rangle$. In general all bulk interactions must be taken into account, including couplings to the graviton and all other modes of the spectrum, as well as a sum over all exchange channels.



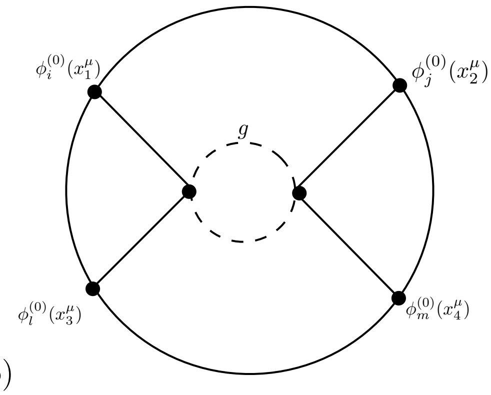


Figure 6.3: Two Witten diagrams contributing to the boundary four-point function $\langle \mathcal{O}_i \mathcal{O}_j \mathcal{O}_l \mathcal{O}_m \rangle$ involving bulk graviton exchange. The first contribution (panel a) is at tree level, the second (panel b) at one loop. In general these contribute to the correlation function at order $\frac{1}{M_{\mathrm{P}}^{2+2L}}$, where L is the number of loops and M_{P} is the Planck mass. For the example of the $\mathcal{N}=4$ theory we have $M_{\mathrm{P}} \sim N$. Of course all bulk fields may run in loops and they may come with their own small coupling parameter (such as λ above). The loop counting proceeds as in ordinary EFT.

6.4 Lorentzian Formalism

As usual we could construct Lorentzian correlation functions by suitable analytic continuations from the Euclidean ones, but for various reasons discussed previously it is useful to have a natively Lorentzian framework for the computation of correlation functions. As we have seen, this is provided by the various incarnations of the two-time formalism. The holographic realization of the contour idea means that we need to find 'infilling' solutions for the various different contours (e.g. the Schwinger-Keldysh contour or the Kadanoff-Baym contour, etc.). As we saw above and in particular from the path-integral point of view, this involves the construction of manifolds that contain both Euclidean and Lorentzian sections, glued together at common boundaries.

In this case the semi-classical saddle point of the gravity action looks like

$$Z\left[\Phi_i \to \phi_i^{(0)}\right] \sim e^{-iS_1 + iS_2 - S_{\mathrm{E}}},$$

where S_{E} represents the contribution from the Euclidean parts of the solution (and may or may not be present, according to the contour we are using) and $S_{1,2}$ are the two Lorentzian parts. The crucial point to appreciate is that the asymptotic structure near each aAdS boundary is exactly the same our previous Euclidean analysis, so long as we replace, in each case, the Laplacian $\Box$ with the d'Alambertian $\Box_{dA}$ in Lorenztian signature. Except for this trivial replacement the asymptotic analysis for each region goes through exactly as before. The reason for this is that all asymptotic coefficients needed to determine the counterterms followed *algebraically* in the previous analysis. Of course, there are potential further subtleties arising from the gluing procedure as well as from additional time-like boundaries that arise at the end of the 'vertical' parts of the contours in the in-in formalism, but once the dust settles one can show that these do not lead to further divergences. The upshot is thus that one adds the usual counterterm action for each part of the contour (and infilling solution) that has an aAdS boundary, which defines a finite renormalized generating functional for Lorenztian correlation functions out of equilibrium. We show an explicit example of such a gluing in Fig. 6.4.

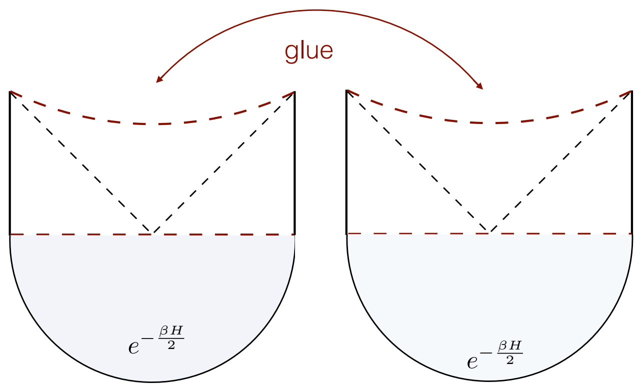


Figure 6.4: An example of a glued bulk manifold. Here we take the AdS black hole (e.g. BTZ to be specific). The two rounded sections each correspond to half the Euclidean black hole (in light gray), each then glued to a Lorentzian section of the black hole at the instant of time symmetry. The manifold is glued together along the dashed (red) lines. The gluing procedure shown here in fact naturally produces the thermal contour with $\sigma = \beta/2$: we have in fact split up the Euclidean evolution into two halves as shown. This is the natural gluing realizing Lorentzian $n-$point functions in the Hartle-Hawking state. It neatly explains why the naive gravity calculation in the Hartle-Hawking state yields he field theory correlations for the contour choice with $\sigma = \beta/2$.

$$W_{\text{ren}}\left[\phi_1^{(0)}, \phi_2^{(0)}\right] := \log Z_{\text{ren}}\left[\phi_1^{(0)}, \phi_2^{(0)}\right] . \tag{6.50}$$

Note that here we only allowed for sources on the Lorentzian parts of the contour. This then leads to well defined renormalized n-point functions

$$iG_{a_1 \dots a_n}(x_1, \dots, x_n) = \frac{1}{i^n} \frac{\delta^n W_{\text{ren}}\left[\phi_1^{(0)}, \phi_2^{(0)}\right]}{\delta\phi_{a_1}^{(0)} \cdots \delta\phi_{a_n}^{(0)}} . \tag{6.51}$$

The solutions in the bulk must now be matched regularly across the various joins in the manifold, reflected in the bulk-to-bulk (6.46) and bulk-to boundary propagators (6.45). In fact these manipulations are standard hailing back to the glory days of black hole thermodynamics (see e.g. Ref. [?] of Chapter 5), and ported in a detailed fashion to the holographic context in [4, 5, 6]. We refer the interested reader there for more details. We have now treated, in some detail the most general framework to calculate arbitrary renormalized n-point functions in holography, generalizing the original recipe for two-point functions of Son and Herzog to arbitrary initial states, including those far from equilibrium. We close with the comment that for most practical application thus far, only low-order correlations (actually one- and two-point functions) near equilibrium (i.e. around a black hole background) are needed. As we reviewed in detail in this case we have a fluctuation-dissipation relation, which allows us to reconstruct the full set of two-point functions from the knowledge of only one Lorentzian asymptotic region. This explains why the majority of calculations in the literature do not involve the more complicated formalism described here. It is hoped that the somewhat complicated treatment in this chapter has served to put this formalism on solid footing, removing the ad-hoc-ness of the usual presentation. It is also hoped that the more adventurous among you, equipped with the full story, embark on an analysis of the many exciting physical results that await to be discovered in the fully non-equilbrium regime where the present framework is indispensable.

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