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Chapter 7

Strongly Coupled Transport

7.1 Transport in 2+1

Transport concerns the study of how physical systems – mostly in the context of condensed matter – respond to external forces via currents. Familiar examples include the conductivity which quantifies how charge is moved in the form of an electric current in response to an external electric field. Similarly heat conductivity quantifies the momentum current in response to an applied heat gradient. Finally, viscosity measures momentum transport as a response to an applied shear. It is fair to say that the most famous result of applied to holography to date is the universal shear-viscosity to entropy ratio first calculated by Policastro, Son and Starinets for the $\mathcal{N} = 4$ SYM theory [1]. In this chapter we will bring some of the technology we developed in previous chapters to bear on a variant of this calculation. In particular we will calculate the shear viscosity of a $2+1$ strongly coupled quantum field theory, dual to a $3+1$ dimensional asymptotically AdS geometry. Since the result will be universal for this class of theories, we do not need to specify exactly which boundary theory we have in mind, but it may still be helpful to note that a concrete top down example exists in the form of the ABJM superconformal field theories, dual to a stack of M2 branes. This calculation was first done in [2], before the advent of the ABJM theories.

Let us recall that the (shear) viscosity is given by a Kubo formula :

$$\eta = \lim_{\omega \to 0} \left[\lim_{q \to 0} \frac{1}{\omega} G^R_{xy,xy}(\omega, q) \right], \tag{7.1}$$

where $G^R_{xy,xy}(\omega, q)$ refers to specific components of the stress tensor two point function,

$$G^R_{\mu\nu,\rho\sigma}(t,\vec{x}) = \Theta(t)\langle[T_{\mu\nu}(t,\vec{x}), T_{\rho\sigma}(0,0)]\rangle$$

in momentum space. Similarly there is a Kubo formula for conductivity

$$\sigma = \lim_{\omega\to 0}\left(\lim_{q\to 0}\frac{1}{\omega}G_{J_xJ_x}(\omega,q)\right)\,, \tag{7.2}$$

where now we compute the retarded current-current two point function

$$G^R_{J_\mu j_\nu}(t,\vec{x}) = \Theta(t)\langle[J_\mu(t,\vec{x}), J_\nu(0,0)]\rangle.$$

Note that in general these are complex quantities, but our main interest here is in their imaginary parts. Referring back to the chapter on linear response the reader can verify that this is equivalent to focusing on the *real* part of the conductivity matrix, Reσ_{ij}. Furthermore, one can consider these quantities as functions of frequency and momentum, but what one usually calls *the* conductivity is the DC conductivity, that is the limit where momentum goes to zero and then frequency goes to zero of the more general object. Similarly for *the* shear viscosity.

7.1.1 Charge Diffusion & Conductivity

We begin with the calculation of charge transport (7.2). A conserved current in the boundary field theory is dual to a U(1) gauge field in the bulk

$$J_\mu \quad \leftrightsquigarrow \quad \text{gauge field } A_a(x^b)$$

$$G^R_{J_\mu J_\nu} \quad \leftrightsquigarrow \quad \text{ingoing modes of } A_a(x^b) \tag{7.3}$$

with action

$$S_{\text{current}} = -\frac{1}{\ell^2}\int d^4x\sqrt{-g}F_{ab}F^{ab}\,. \tag{7.4}$$

This is placed in the AdS_4 black hole background

$$ds^2 = \frac{\ell^2}{z^2}\left(-f(z)dt^2 + \frac{dz^2}{f(z)} + d\vec{x}\cdot d\vec{x}\right) \tag{7.5}$$

with $f = 1 - \left(\frac{z}{z_h}\right)^3$. There is a planar horizon at coordinate value $z = z_h$, while the metric asymptotes to Poincaré AdS. For this calculation we take the bottom-up perspective. However, a concrete top-down approach could be built using M2 branes, as has indeed been the perspective in the original paper [2]. Firstly we fix the axial gauge $A_z = 0$, which is quite convenient for holographic calculations.

However, it leaves the residual gauge freedom $A_a \to A_a + \partial_a \Lambda(x^\mu)$, where Λ does not depend on z. This is like a gauge transformation in the boundary theory, and in order to fix it, one must impose a further gauge constraint at a single constant z surface. It is often convenient to use the horizon to do this, but the boundary, or indeed any other constant z surface would do just as well.

Let us work in Fourier space:

$$A_\mu(z, x^\mu) = \int \frac{d^3q}{(2\pi)^3} e^{-i\omega t + iqy} A_\mu(\omega, q), \tag{7.6}$$

where we have oriented $\vec{q} = q\hat{e}_y$ without loss of generality, a reflection of the $SO(2)$ symmetry of the background.

The equations of motion, i.e. the covariant Maxwell equations in the black-hole background, read

$$\mathfrak{w} A_t' + \mathfrak{q} f A_y' = 0 \tag{7.7}$$

$$A_t'' - \frac{1}{f}\left(\mathfrak{q}^2 A_t + \mathfrak{w}\mathfrak{q} A_y\right) = 0 \tag{7.8}$$

$$A_x'' + \frac{f'}{f} A_x' - \frac{1}{f}\left(\mathfrak{q}^2 - \frac{\mathfrak{w}^2}{f}\right) A_x = 0 \tag{7.9}$$

$$A_y'' + \frac{f'}{f} A_y' + \frac{\mathfrak{w}^2}{f^2} A_y + \frac{\mathfrak{w}\mathfrak{q}}{f^2} A_t = 0. \tag{7.10}$$

Comments

1. The first equation, (7.7), is a constraint expressing the current Ward identity

   $$\partial_\mu \langle J^\mu \rangle = 0\,.$$

   This is a general feature: bulk local symmetries – in this case U(1) – lead to boundary Ward identities expressing the conservation of the dual current. Of course if we add sources, the Ward identity will become the continuity equation.

2. For convenience we have gone to dimensionless variables

   $$(\omega, q) \to (\mathfrak{w}, \mathfrak{q}) =: z_h(\omega, q)$$

   corresponding to the rescaled radial coordinate $z \to \frac{z}{z_h}$ Since $T \sim 1/z_h$ this is saying that we measure all energies in units of temperature from now on.

3. The equation for A_x decoupled from A_y and A_t. In fact, this is a consequence of symmetry: A_x is the transverse (to the momentum vector) component of the gauge field, and so by parity it cannot mix with the remaining ones. A similar analysis will turn out to be very helpful for the later case of the shear viscosity where we will have to deal in addition with the tensor modes coming from the metric perturbation. Here it means we can distinguish two sectors, the longitudinal A_y, A_t sector, and the transverse A_x sector. These correspond to different physical processes in the boundary theory, as we will see presently.

We first solve the A_y, A_t sector. First, solving (7.8) for A_y gives

$$A_y = \frac{f}{\mathfrak{q}\mathfrak{w}} A_t'' - \frac{\mathfrak{q}}{\mathfrak{w}} A_t \,. \tag{7.11}$$

Now, substituting this into (7.10), we get a single third-order differential equation,

$$A_t''' + \frac{f'}{f} A_t'' + \frac{1}{f}\left(\frac{\mathfrak{w}^2}{f} - \mathfrak{q}^2\right) A_t' = 0, \tag{7.12}$$

which is a second order ODE for $\phi = A_t'$ with no analytically known solution. However, we know that the Kubo formula for the transport coefficient only needs $\mathfrak{w}, \mathfrak{q} \ll 1$, so we can develop an expansion in small $(\mathfrak{w}, \mathfrak{q})$. Furthermore we are interested in the retarded correlation function, so we look for the ingoing mode at the horizon

$$\phi \sim (1-z)^{-\frac{i\mathfrak{w}}{3}} F(z), \tag{7.13}$$

then $F(z)$ can be developed as as series in small frequency and momentum

$$F(z) = F_{(0)} + \mathfrak{w} F_{(1,0)} + \mathfrak{q}^2 F_{(0,2)} + \mathfrak{w}^2 F_{(2,0)} + \cdots \tag{7.14}$$

A somewhat tedious, but straightforward, calculation[1] gives the first few terms in this expansion

[1] Seek the help of Mathematica, Maple, Dr Valdez.

$$F_{(0)} = C$$
$$F_{(1,0)} = -\frac{C}{6}\left[2\sqrt{3}\tan^{-1}\left(\frac{1+2z}{\sqrt{3}}\right) + \log\left(\frac{1-z^3}{1-z}\right)\right]$$
$$F_{(0,2)} = -\frac{2C}{\sqrt{3}}\tan^{-1}\left(\frac{1+2z}{\sqrt{3}}\right)$$

for an arbitrary constant C, which we will determine shortly. In order to uniquely determine these we had to require: 1) regularity at the horizon order by order, and 2) that only the leading order contains the homogeneous piece. We can now use (7.11) to determine the constant C in terms of the boundary data of the longitudinal vector modes,

$$C = \frac{\mathfrak{q}^2 A_t^{(0)} + \mathfrak{w}\mathfrak{q} A_y^{(0)}}{i\mathfrak{w} - \mathfrak{q}^2}, \tag{7.15}$$

which is gauge invariant under residual gauge transformations, which is easy to see as the denominator is the Fourier transform of the field strength component $F_{yt} = (\partial_y A_t - \partial_t A_y)$. We thus have

$$A_t = A_t^{(0)} + z\,C + O(z^2) \tag{7.16}$$
$$A_y = A_y^{(0)} - z\frac{\mathfrak{w}}{\mathfrak{q}}C + O(z^2) \tag{7.17}$$

Note:

1. $A_{t,y}^{(0)}$ are the boundary data for the fields in the longitudinal sector, that is the equivalent of $\phi_{(0)}$ in previous sections. They have the interpretation as sources for the longitudinal components of the current in the boundary.

2. The terms of order $O(z)$ are *not* determined by near-boundary analysis (the analog of the coefficient $\phi_{(2\Delta-d)}$ in our previous analysis for a scalar operator of dimension $\Delta = 2$, $d = 3$). However, we have been able to determine them (7.15) by finding full bulk solution, linking the solution with ingoing boundary conditions in the infrared to the UV data.

We can now read off the components of the retarded correlation function in the longitudinal sector, namely

$$G^R_{tt}(\mathfrak{w},\mathfrak{q}) = \mathcal{N}\frac{\mathfrak{q}^2}{i\mathfrak{w}-D\mathfrak{q}^2} \tag{7.18}$$

$$G^R_{yy}(\mathfrak{w},\mathfrak{q}) = \mathcal{N}\frac{\mathfrak{w}^2}{i\mathfrak{w}-D\mathfrak{q}^2} \tag{7.19}$$

$$G^R_{ty}(\mathfrak{w},\mathfrak{q}) = \mathcal{N}\frac{\mathfrak{q}\mathfrak{w}}{i\mathfrak{w}-D\mathfrak{q}^2}\,, \tag{7.20}$$

with $\mathcal{N}=\frac{3}{4e^2(2\pi)^3}$. Noteworthy is the location of the pole, as a function of frequency and momentum

$$i\mathfrak{w} = D\mathfrak{q}^2\,, \tag{7.21}$$

where, reinstating units, we find $D = z_h = \frac{3}{4\pi T}$. A relation between frequency and momentum of a pole is known as a dispersion relation, and the form (7.21) is referred to a diffusive pole. This is intuitive, as it is basically the diffusion equation

$$\partial_t n = D\nabla^2 n \tag{7.22}$$

in momentum space. Here n is the density of the diffusing quantity, be it charge or the concentration of dye dropped in a container of water. The physics of the longitudinal sector is thus that of charge diffusion, and using our Lorentzian holographic techniques, we have determined the charge diffusion constant of a strongly coupled $2+1$ quantum field theory. Performing this calculation using conventional techniques (essentially perturbation theory) would be no small feat, but here it was a relatively easy calculation illustrating the power of holography.

In order to find the actual electric conductivity we should solve the equation for A_x, the transverse component. Actually this is now almost trivial, as we have done all the hard work already: the resulting differential equations are of exactly the same form as the ones we solve above. We start again by peeling off the ingoing behavior at the horizon, and posing

$$A_x = (1-z)^{-\frac{i\mathfrak{w}}{3}}F(z), \tag{7.23}$$

where

$$F(z) = F_{(0)} + \mathfrak{w}F_{(1,0)} + \mathfrak{q}^2F_{(0,2)} + \cdots$$

with coefficient functions $F_{(i,j)}$ exactly as above. Expanding $A_x(z,\mathfrak{w},\mathfrak{q})$ near $z=0$, we find

$$F(z) = \frac{C}{18}(18 + i\sqrt{3}\pi\mathfrak{w} - 2\sqrt{3}\mathfrak{q}^2\pi) + C\left(\frac{2i\mathfrak{w}}{3} - \mathfrak{q}^2\right)z + \cdots \tag{7.24}$$

Proceeding as usual to extract the correlator resuls in

$$G^R_{xx}(\mathfrak{w}, \mathfrak{q}) = -\mathcal{N}(i\mathfrak{w} - D\mathfrak{q}^2), \tag{7.25}$$

From the Kubo formula we learn that the coefficient of $\mathfrak{w}$ gives the conductivity, in the present case

$$\sigma^{(2+1)}_{ij} = \delta_{ij}\mathcal{N} \tag{7.26}$$

Comments

1. If we match parameters to the field theory via the top-down M2 brane picture, we find $\mathcal{N} \sim \frac{1}{\ell^2} \sim N^{3/2}$ for $M2$ brane theory. This is an echo of the famous $N^{3/2}$ scaling of the free energy of the M2 brane theory.

2. In summary, our results for charge transport in a strongly coupled $2+1$ dimensional quantum field theory are

$$\begin{aligned} \sigma_{ij} &= \mathcal{N}\delta_{ij} \\ D &= \frac{3}{2\pi T} \end{aligned}$$

We now proceed to the momentum sector, which involves a calculation very similar to the above, but slightly more technically complicated due to the involvement of tensor fluctuations associated with the metric. The general procedure is the same, that is we identify sectors of linear perturbations which we then solve in a low frequency low momentum expansion with ingoing boundary conditions at the horizon. Taking the appropriate limits we extract the transport coefficients.

7.1.2 Momentum Diffusion & Viscosity

As we saw in Chapter 5 , the shear viscosity quantifies the response of a system to a velocity gradient. This results, at the microscopic level, from transport of momentum along the gradient. Thus we have

$$\text{momentum transport} \longleftrightarrow \text{viscosity}$$

The momentum current in a quantum field theory is given by the spatial component T_{0i} of the stress tensor, so we need to consider the stress energy 2-point function of the dual quantum field theory. This is encoded holographically via the bulk metric and its fluctuations

$$T_{\mu\nu}(t,\vec{x}) \quad \leftrightsquigarrow \quad \text{metric mode } g_{ab}(z,x^{\mu})$$

$$G^R_{T_{\mu\nu},T_{\rho\sigma}}(t,\vec{x}) \quad \leftrightsquigarrow \quad \text{ingoing mode of } \delta g_{ab} := h_{ab} \tag{7.27}$$

where we choose ingoing boundary conditions because, according to the Kubo formula, we need to determine the retarded correlation function. Because the metric fluctuation is a rank two tensor, we will do well to organise our calculation, using the fact that h_{ab} decomposes into sectors according to the symmetry of the background metric. Again we make the gauge choice $h_{za} = 0$, and work in Fourier space

$$h_{\mu\nu}(z,x^{\mu}) = \int \frac{d^3q}{(2\pi)^3} e^{-i\omega t + iqy} h_{\mu\nu}(\omega,q), \tag{7.28}$$

where the momentum has been oriented as $\vec{q} = q\hat{e}_y$. The radial gauge again does not fully fix the freedom and there will be residual diffeomorphisms. Since the viscosity is determined by the specific components, $\langle T_{xy}, T_{xy}\rangle$, of the stress-stress two-point function, which in turn is dual to the h_{xy} fluctuation, we should keep all modes that mix with h_{xy}. We can use parity along the transverse direction $(x \to -x)$ to classify these sectors. The components behave in the following way under parity

$$\begin{aligned} h_{xy} &: \text{odd} \\ h_{xx} &: \text{even} \\ h_{yy} &: \text{even} \\ h_{tx} &: \text{odd}, \end{aligned}$$

implying that only h_{tx} can mix with h_{xy}. It is convenient to define

$$\begin{aligned} h^x_y &:= e^{-i\omega t + iqy} H_y(z) \\ h^x_t &:= e^{-i\omega t + iqy} H_t(z)\,. \end{aligned}$$

With these defintions, the linearized Einstein equations give

$$H_t'' - \frac{2}{z}H_t' - \frac{1}{f}(\omega q H_y + q^2 H_t) = 0 \tag{7.29}$$

$$H_y'' + \frac{f-3}{zf}H_y' + \frac{1}{f^2}(\omega^2 H_y + q\omega H_t) = 0 \tag{7.30}$$

$$\omega H_t' + fqH_y' = 0\,, \tag{7.31}$$

which are very similar in structure to what we saw in the case of charge transport. Proceeding as in that case, one finds, in terms of the dimensionless variables $\mathfrak{w}, \mathfrak{q}$

$$H_t''' + \frac{f-3}{xf}H_t'' + \left[\frac{1}{f^2}(\mathfrak{w}^2 - f\mathfrak{q}^2) - \frac{4}{z^2} + \frac{6}{z^2 f}\right]H_t' = 0. \tag{7.32}$$

Taking $H_t' = \Phi$ and stripping off the ingoing behavior at the horizon

$$\Phi = (1-z)^{-\frac{i\omega}{3}}F(z)$$

the expansion for $\mathfrak{w}, \mathfrak{q} \ll 1$ becomes

$$F(z) = C\left\{z^2 + i\omega\left[z(z-1) + \frac{z^2}{6}f_1(z) - \frac{z^2}{\sqrt{3}}f_2(z)\right] + \frac{q^2}{3}z(1-z)\right\}, \tag{7.33}$$

where

$$f_1(z) = \log\left(\frac{1}{3}(1+z+z^2)\right)$$

$$f_2(z) = \tan^{-1}\left(\frac{1+z^2}{\sqrt{3}}\right). \tag{7.34}$$

Exactly as in the conductivity case, we consider the constraint equation (7.29) near $z = 0$ to determine the constant

$$C = \frac{\mathfrak{q}\mathfrak{w}H_y^{(0)} + \mathfrak{q}^2 H_t^{(0)}}{i\mathfrak{w} - \frac{1}{3}\mathfrak{q}^2}. \tag{7.35}$$

This time let us read off the two-point functions from the on-shell action, which looks like

$$S_{\text{on-shell}} = \frac{2^{5/2}\pi^2}{3^4}N^{3/2}T^3\int dz\, d^3x \frac{1}{z^2}\left((H_t')^2 - f(H_x')^2 + \cdots\right) \tag{7.36}$$

Thus we have the correlators

$$G_{ty,ty} = \mathcal{N}\frac{\mathfrak{q}^2}{i\mathfrak{w} - D\mathfrak{q}^2} \tag{7.37}$$

$$G_{ty,xy} = -\mathcal{N}\frac{\mathfrak{w}\mathfrak{q}}{i\mathfrak{w} - D\mathfrak{q}^2} \tag{7.38}$$

$$G_{xy,xy} = \mathcal{N}\frac{\mathfrak{w}^2}{i\mathfrak{w} - D\mathfrak{q}^2} \tag{7.39}$$

with

$$\mathcal{N} = \frac{2^{2/3}\pi N^{3/2}T^2}{3^3}\,, \qquad D = \frac{1}{4\pi T}\,. \tag{7.40}$$

The Kubo formula then gives the shear viscosity

$$\eta = \frac{2^{3/2\pi}}{3^3}N^{3/2}T^2. \tag{7.41}$$

We now recall from previous chapters our result for the entropy density of the planar black hole in AdS, which in field-theory units reads

$$s = \frac{9\sqrt{2}\pi^2 N^{3/2}}{27}T^2 \tag{7.42}$$

Hence the ratio of shear viscosity to entropy density comes out to be

$$\frac{\eta}{s} = \frac{2^{2/3}\pi\, 27}{3^3 8\sqrt{2}\pi^2} = \frac{1}{4\pi} \tag{7.43}$$

which is the famous result for $\mathcal{N} = 4$ SYM in $3+1$ and evidently also holds in $3+1$ dimensional strongly coupled field theory, e.g. the ABJM theory.

Bibliography

[1] G. Policastro, D. T. Son and A. O. Starinets, *"The Shear Viscosity of Strongly Coupled $\mathcal{N} = 4$ Supersymmetric Yang-Mills Plasma,"* Phys. Rev. Lett. **87** (2001) 081601 doi:10.1103/PhysRevLett.87.081601 [hep-th/0104066].

[2] C. P. Herzog, *"The Hydrodynamics of M Theory,"* JHEP **0212** (2002) 026 doi:10.1088/1126-6708/2002/12/026 [hep-th/0210126].