Solvay Lectures
On Applied Holography

CERN, winter 2016

Julian Sonner

UNIVERSITÉ DE GENÈVE

FACULTÉ DES SCIENCES
Département de physique théorique

Preamble

These notes arose out of a course given by the author at CERN in 2015 and 2016 as part of the Solvay school for beginning PhD students from all over Europe. The notes are targeted at an audience - such as that of the school - which already has some familiarity with holography, for example through a first course at beginning PhD level. It is the aim of the present course to explain a selection of the tools used in this active branch of research with a strong emphasis on real-time physics, which is developed both in field theory and from the holographic point of view. Real-time physics is at the forefront of research in holography these days, both in order to target applications to non-equilbrium physics, but also because it is turning out to be an indispensable tool for understanding the duality at a fundamental level and to make it work for the purposes of quantum gravity itself.

These notes are aimed to give a more in-depth technical treatment of the subject in order to equip the reader with the necessary tools to carry out research in this direction. There exist several excellent lectures and books which would serve well to complement this course, such as

- Sean Hartnoll: "'*Lectures on holographic methods for condensed matter physics*"
- John McGreevy: " *Holographic duality with a view towards many-body physics*"
- Ammon & Erdmenger: "*Gauge/Gravity Duality: Foundations and Applications*"
- Zaanen, Sun, Liu, Schalm: "*Holographic Duality in Condensed Matter Physics*"

I would like to thank my student Manuel Vielma for typing up my handwritten notes from the first time I gave the course in 2015. These texts served as the basis for the present form of the lectures. I would also like to thank the students at the Solvay school in 2015 and 2016 for asking stimulating questions and for pointing out various typos to help improve these notes. Any remaining mistakes and typos remain the sole responsibility of the author.

Julian Sonner, Geneva (November 2016)

Contents

Chapter 1

Introduction and Preview

1.1 Introduction

The reader is assumed to already have some familiarity with the basic ideas and principles of AdS/CFT, also often known as gauge-gravity duality, or simply holography. The latter two names are preferred, in particular in the context of applied AdS/CFT as they do not imply a restriction to anti-de Sitter space or conformal field theories. In fact, holography is not restricted to these, nor does holographic duality happen solely in theories with gauge interactions. A lot of research these days concerns itself with the questions of what properties a quantum field theory should have in order for it to be dual to a gravitational theory. This will not be the focus of these lectures, rather we will take it as a given that there exist certain dual pairs – as we indeed know to be the case – and then use these to gain insights into strongly coupled physics.

We will develop tools that allow us to ask the question what AdS/CFT tells us about physical phenomena. Before we proceed we should therefore remind ourselves of the basic setup of that duality, for the time being restricted to the case of gauge fields and Einstein (-like) gravity in asymptotically AdS spaces. In this case we have the following dual relationships of parameters.

$$\begin{pmatrix}\text{gauge theory } SU(N) \\ \text{in } d \text{ spacetime dimensions}\end{pmatrix} \xleftrightarrow{\text{dual to}} \begin{pmatrix}\text{quantum gravity on AdS} \\ \text{in } d+1 \text{ dimensions}\end{pmatrix}$$

$$1/N^2 \qquad\qquad G_N$$

$$g_{\mathrm{YM}}^2 N \, (\sim g_s N) \qquad\qquad (\alpha')^2 \sim \ell_s^4$$

$$\lambda \qquad\qquad \ell^4/\ell_s^4$$

Here g_{YM} is the gauge coupling, while N is the number of colors of the gauge group. The specific power of N in the relationship between the Newton constant and the rank of the gauge group depends on the details of the dual pair. Here we have used the power appropriate for the most famous example between the $\mathcal{N} = 4$ theory and type IIB strings. It is customary to denote the combination $g_{YM}^2 N$, the so-called 't Hooft coupling, by the letter λ. We have denoted the AdS radius by ℓ and the string length by the symbol ℓ_s.

The duality is defined for any value of N and λ. For moderate values of N and λ we are confronted with a bulk string theory for which quantum corrections are significant – more on this in the next section. From the bulk gravity point of view, this is a challenging setting and much is still left to be understood about it. On the other hand, so long as λ is small enough so as to keep the field theory in the perturbative regime, calculations on that side become feasible. It is an interesting and active field of research to explore what can be learnt about quantum gravity from this approach. Since the main focus of these lectures is on the study of field theory through the lens of dual gravity we will focus on a different regime of parameters. As we see from the relations above, the limit $N \to \infty$ suppresses quantum gravity corrections. This still leaves us with classical string theory in the bulk, so often a further limit $\lambda \to \infty$ is taken, or equivalently $\ell_s \to 0$. This corresponds to sending the string tension to infinity, causing the strings to collapse down to point particles. The resulting gravitational dual is then described by point-particle excitations above a classical gravity background.

To summarize, in applied holography we mainly consider the field theory in the regime defined by

1. The planar limit, $N \to \infty$: no quantum gravity corrections.
2. The strong coupling limit, $\lambda \to \infty$: no stringy corrections.

From the bulk point of view this puts us in a computationally accessible regime, which, astonishingly, is dual to the most inaccessible regime of the field theory, namely that of ultra strong coupling. The idea is then to learn about ultra-strong coupling field theory physics by studying a dual problem in classical gravity. Physics is full of interesting systems which are strongly coupled and thus largely inaccessible to conventional methods. Examples are found in the description of the strong interactions (QCD) as well as numerous systems in the realm of strongly-correlated matter. A particularly well-known such example is furnished by the strongly correlated class of materials exhibiting high-T_C superconductivity.

A Word of Caution

We should keep in mind that none of the systems we know about in nature satisfy the first of our requirements, namely that of planarity ($N \gg 1$) essentially the requirement that they have a very large number of local degrees of freedom. We can take two attitudes towards this.

Large-N Expansion

Firstly we can take the point of view that the solution of the infinite-N theory is a good starting point for an approximate treatment of the system in powers of $1/N$. The non-trivial insight of holography is to recognize that the infinite N theory is described by gravity in one higher dimension, rather than a conventional saddle point within field theory. Note that the saddle point at infinite N still includes an infinite set of planar diagrams i.e. the expansion in the 't Hooft coupling. This is very challenging and the infinite-N solution has still not been found for QCD for example. Having the benefit of the holographic large-N saddle point in terms of bulk gravity, still leaves us with the challenge to compute the $1/N$ corrections from the gravity point of view, a point we shall return to below.

Focus on Universal Quantities

The second point of view is to restrict our attention to observables and mechanisms which display universality across systems at strong coupling. Such universal physics will then hold true for every theory with a holographic dual, including the unknown duals of fully realistic systems, should they exist. This way we aim to extract more general lessons for strong coupling physics, using holography as a rare glimpse into ultra-strong coupling physics, not afforded by other methods.

Before concluding this introduction, let us mention that besides the gauge theory / string theory dual pairs discussed above, there are other examples of holographic duality, for example

$$\begin{pmatrix}\text{vector-like } N\text{-component}\\ \text{QFTs}\\ \text{(e.g. } O(N) \text{ model)}\end{pmatrix} \xleftrightarrow{\text{dual to}} \begin{pmatrix}\text{higher-spin}\\ \text{`quantum gravity' theories}\\ \text{in AdS}\end{pmatrix} \tag{1.1}$$

Here the boundary theory is the well-understood class of large$-N$ vector models while the bulk duals are the rather more complicated than gravity theories which contain an infinite tower of massless higher-spin fields, above the spin-2 graviton.

Very recently a new set of examples of holographic duality has been proposed, relating

$$\begin{pmatrix}\text{quenched disorder}\\ \text{many-body QM}\end{pmatrix} \xleftrightarrow{\text{dual to}} \begin{pmatrix}\text{AdS}_2\\ \text{`quantum gravity' theories}\end{pmatrix} \tag{1.2}$$

These latter two examples involve much simpler field theories, but the trade-off is that the bulk side is less familiar than ordinary Einstein gravity. It is an open question whether these are interesting in the context of applied AdS/CFT. Before we move on to developing the machinery allowing us to extract physically interesting quantities using holographic duality, it will turn out to be helpful to spend some time reviewing where these dualities come from and in what contexts they are most well understood. As a first step, let us discuss the modern point of view on the bulk quantum gravity, via effective field theory.

1.2 Gravity as an Effective Field Theory

During the course of these lectures we will often encounter statements such as 'let us calculate the gravity partition function on a black hole background', or other such statements that are really located in the realm of quantum gravity. As we have already mentioned, in the majority of well-understood cases[1], the bulk dual is a string theory and so a UV complete theory of quantum gravity. In practice however, most bulk calculations are not performed using the full machinery of string theory, in fact often one finds no reference to anything but ordinary gravity in the presence of certain matter fields, especially in the so-called bottom-up perspective (more about this later). But general relativity has a famously rocky relationship with quantum mechanics, so how do we proceed concretely when we perform bulk gravity calculations?

The answer lies in the framework of effective field theory (EFT). This point of view gives us a formalism to consistently deal with quantum gravity at low energies without being sensitive to the problems arising in the UV. These problems, of course, are rooted in the non-renormalizability of GR as a quantum field theory. These troubles are cured in string theory, but the EFT approach allows us to be, in some sense, agnostic about the details of the eventual UV completion and still obtain non-trivial results at low energies. We shall not give a comprehensive treatment and the reader who is interested to learn more is encouraged to consult the classic review by Donoghue [1]. Our presentation here borrows heavily from [2], but note that the discussion there focusses mostly on EFT around flat space.

[1]Perhaps even in all consistent cases.

The Derivative Expansion

The philosophy of EFT is to first identify the symmetries of the problem and then write down the most general Lagrangian compatible with those symmetries, arranged in an expansion in ascending numbers of derivatives. The idea underlying this organizational structure is that some new physics, necessary to render the theory UV complete must enter at some scale $M_{\rm new}$ and will appear with coefficients proportional to positive powers of $E/M_{\rm new}$. Its effects will thus be small at low energies compared to $M_{\rm new}$, i.e. then $E/M_{\rm new} \ll 1$. Thus a truncation at a given derivative level will give answers accurate to a given power in $E/M_{\rm new}$, furnishing a systematic way of approximating a given physical quantity. In this sense it is consistent to truncate to a given level in the derivative expansion, essentially treating loop and renormalization effects only for the set of operators within the truncation. Higher-order operators will be suppressed at low energies by positive powers of $1/M_{\rm new}$, so we are justified to approximately treat this sector effectively as if it were a renormalizable theory.

For the case of general relativity the most important symmetry principle is general covariance. The first few terms in an expansion in $\nabla_a \sim p_a$, compatible with general covariance, are

$$S = \frac{1}{2\kappa^2}\int \sqrt{-g}\left(-2\Lambda + R + c_1 R^2 + c_2 R_{ab}R^{ab} + c_3 R_{abcd}F^{abcd} + \cdots\right) d^D x\,, \quad (1.3)$$

where $\kappa^2 = 8\pi G_N$. If we wish to add matter, we arrange it similarly in a derivative expansion and keep terms to the same order as the gravity part. In EFT one expects the coefficients of all allowed terms to be $\mathcal{O}(1)$ numbers times the appropriate power of $1/M_{\rm new}$, whose precise values are fixed by matching to experiment[2] or, for the case of holography, by the requirements of the dual theory. In general relativity the predictions of effective field theory become non-sensical (e.g. non-unitary) for processes with energies around the Planck scale, the natural scale of gravity defined by the dimensionful coupling

$$G_N = M_{\rm P}^{2-D}\,. \quad (1.4)$$

[2]Thinking about gravity in the real world, this is saying that the cosmological constant (or better the inverse Hubble radius) is $1/\ell_{\rm P}$, i.e. the universe is of size $\mathcal{O}(1)$ times the Planck length. But that's another story.

This implies that we must have new physics enter at the Planck scale at the latest. Actually the requirement is only that $M_{\text{new}} < M_{\text{P}}$, that is we could have new physics come in at a scale lower than the Planck scale. In fact, this is the case in string theory, with important implications for holography. In string theory, we have

$$c_{1,2,3} \sim \frac{1}{M_{\text{new}}^2} \sim \ell_s^2\,, \tag{1.5}$$

where ℓ_s is the string length. These corrections come from the fact that string theory is fundamentally not a theory of point particles but of one-dimensional objects and when the typical energy of a process is of the order of the inverse length of a string this leads to corrections, parametrized here by $c_{1,2,3}$. In summary the EFT expansion in string theory, and thus for many of our holographic examples, is a double expansion in E/M_{P} and E/M_{new}. As summarized at the very beginning of this chapter this is encoded in the dual field theory which is a double expansion in $1/N \sim G_N$ (to the appropriate power) and the ‘t Hooft coupling $\lambda \sim 1/\alpha'$ (to the appropriate power). So long as N and λ are large we may therefore treat the bulk theory as an EFT up to a given level in the derivative expansion and compute loop diagrams as if the theory were renormalizable. This answers the question of what we mean by ‘calculating the partition function of gravity in a black hole background’ and other similar operations. Let us conclude this section by making some remarks about practical matters, such as loops in this EFT.

The Loop Expansion

Most computations in holography follow the general EFT scheme, namely that we expand in metric perturbations about a given background

$$g_{ab} = \overline{g}_{ab} + \kappa h_{ab}\,. \tag{1.6}$$

The factor of $\kappa \sim \sqrt{G_N}$ in front of the metric perturbation is inserted to insure that the kinetic term for metric fluctuations is canonically normalized. In holography we are most interested in the EFT of asymptotically AdS spaces, which we will turn to in a moment. In preparation, let us consider the simpler case of flat space first. Suppressing all indices, the action, expanded around flat space has the form

$$S \sim \int \left(\partial h \partial h + \frac{1}{M_{\mathrm{P}}^{\nu}} \left(h \partial h \partial h + \cdots \right) + \frac{1}{M_{\mathrm{new}}^{2}} \left(\partial^2 h \partial^2 h + \frac{1}{M_{\mathrm{P}}^{\nu}} h \partial^2 h \partial^2 h \right) \right) d^D x$$

where $\nu = \frac{D-2}{2}$. For the simple case of only one scale, that is $M_{\mathrm{P}} = M_{\mathrm{new}}$, amplitudes are organised into an expansion in

$$\left(\frac{E^{2\nu}}{M_{\mathrm{P}}^{2\nu}} \right)^{1+L} , \tag{1.7}$$

where L is the number of loops. Thus, from a dual field theory point of view, each bulk loop diagram adds a factor of $1/N$ (to the appropriate power). We will return to this in a later chapter, when we discuss Witten diagrams. From Eq. (1.7) we learn that the EFT becomes strongly coupled at energies $E \sim M_{\mathrm{P}}$ indicating its own breakdown and the need to include new UV degrees of freedom. On the other hand for energies below this scale the theory is well under control, including arbitrary loop corrections. The actual situation is a little more complicated, since $M_{\mathrm{new}} \neq M_{\mathrm{P}}$ and so the strong coupling scale arises at an energy

$$E \sim M_{\mathrm{P}}^{x} M_{\mathrm{new}}^{1-x} \tag{1.8}$$

for some x between zero and one. This means that the strong-coupling regime begins somewhere between the scales M_{new} and M_{P}.

Let us now turn to the structure in asymptotically AdS spaces. The story is not much different, except that we now have a non-zero value of the background curvature

$$R \sim 1/\ell^2 \sim M_{\mathrm{AdS}}^2 . \tag{1.9}$$

This means that the derivative expansion has additional structure. Expanded around AdS_D the action takes the form

$$\begin{aligned} S \sim \int \sqrt{-\bar{g}} \; \Bigg(& \partial h \partial h + \frac{1}{M_{\mathrm{P}}^{\nu}} h \partial h \partial h \left(1 + \frac{M_{\mathrm{AdS}}^2}{M_{\mathrm{new}}^2} \mathcal{O}(1) + \cdots \right) \\ & + \frac{1}{M_{\mathrm{new}}^{2}} \left(\partial^2 h \partial^2 h + \frac{1}{M_{\mathrm{P}}^{\nu}} h \partial^2 h \partial^2 h \right) \Bigg) d^D x \, , \end{aligned} \tag{1.10}$$

where each of the terms in the second line also have corrections in ascending powers of $\frac{M_{\rm AdS}^2}{M_{\rm new}^2}$ times $\mathcal{O}(1)$ numbers which we do not show. Hence we have the same EFT structure as in flat space with the additional feature that each term comes with an expansion in $\frac{M_{\rm AdS}^2}{M_{\rm new}^2} \sim \frac{\ell_s^2}{\ell^2} \sim 1/\sqrt{\lambda}$.

The important lesson of this section is that the EFT structure of the bulk gravity theory is what allows us to compute quantum gravity partition functions using essentially a quantized form of general relativity. This is what is done in most practical holographic applications, and the point of view we adopt in these lectures. It is good to know – and sometimes essential – that there exists a UV complete superstructure, in the form of string theory which would allow us to beyond the EFT point of view. In practice this is very hard and one more often sees the point of view adopted that the true quantum gravity regime is best described via the boundary field theory itself. Let us now delve deeper into the subject and start working with actual examples of holographic dualities.

1.3 Top-Down vs. Bottom-Up

In many cases, in principle, we have a precise duality relation. The first and most prominent example is given bt the $\mathcal{N} = 4$ SYM theory in $d = 4$ with $psu(2, 2|4)$ superconformal algebra. Its action follows from the Lagrangian

$$\mathcal{L} = \frac{1}{4}\mathrm{Tr}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\mathrm{Tr}D_\mu\Phi_n D^\mu\Phi^n - \frac{1}{4}g^2\mathrm{Tr}\left[\Phi^m, \Phi^n\right]\left[\Phi_m, \Phi_n\right] + \text{fermions}\,, \quad (1.11)$$

where the trace is over the adjoint representation of the gauge group $SU(N)$. The theory has a global $SU(4) \equiv SO(6)$ internal symmetry under which the scalar fields Φ^a transform as the fundamental vector representation of $SO(6)$. It is often helpful to use the fact that the Lagrangian can be obtained as a dimensional reduction of the ten-dimensional minimally supersymmetric Yang Mills theory with gauge group $SU(N)$.

This theory is dual to type IIB string theory on $\mathrm{AdS}_5\times\mathrm{S}^5$. As we already saw, our main interest in this set of lectures is on the planar limit at strong 't Hooft coupling. In that case the string theory reduces to its IIB supergravity approximation (see previous section). Since we are mainly interested in solutions of ten dimensional supergravity of the form $\mathrm{AdS}_5\times\mathrm{S}^5$, or some deformation of the five sphere, it is more convenient to work instead with a consistent truncation of the ten dimensional theory which allows us to determine the relevant gravity solutions in terms of a five dimensional theory defined on AdS_5. In the present case this theory is given by the $\mathcal{N}=8$ five dimensional gauged $SU(4)$ supergravity with supergroup $PSU\,(2,2|4)$. Having a more precise statement of the duality in hand, we can be more concrete about what kind of quantities correspond to each other on the two sides. For example between the $\mathcal{N}=4$ Yang Mills theory (1.11) and the five dimensional gauged supergravity, one can establish that the global symmetries match up

$$\begin{aligned} SU(4)_R \quad &\longleftrightarrow \quad SU(4)_{\text{gauged}} \\ psu(2,2|4)\text{ superconformal} \quad &\longleftrightarrow \quad psu(2,2|4)\text{ AdS supergroup} \end{aligned} \tag{1.12}$$

A more fine-grained check of the duality is encapsulated in the comparison of the spectra of the two theories, which satisfy

$$\begin{gathered} \text{spectrum of BPS states in } \mathcal{N}=4\text{ SYM} \\ \updownarrow \\ \text{spectrum of SUGRA Kaluza-Klein modes on } \mathrm{AdS}_5\times\mathrm{S}^5 \end{gathered}$$

This is all to say that in the present case there exists a precise match between detailed symmetries, spectra and in fact correlation functions (more on this later). These are a set of non-trivial tests giving one confidence in the duality, but they fall of course short of a proof.

The duality relation we just discussed in some detail was originally deduced by thinking about the properties of D3 branes in IIB string theory. Its validity has first been conjectured in a seminal paper by Juan Maldacena in 1997 [1]. To this day it has not been proven, but more and more precise tests, such as the above, both perturbative and non-perturbative in nature, convince us with a high degree of certainty of its general validity. Since then more examples of dual pairs have been proposed and put to the test. For now we contend ourselves with naming but one more example, that of the three-dimensional so-called ABJM theory, a

Chern-Simons theory, dual to a class of M-theory compactifications down to AdS_4:

$$\mathrm{ABJM}_{3D} \xleftrightarrow{\text{dual to}} \mathrm{AdS}_4 \times \mathrm{S}^7/\mathbb{Z}_k \tag{1.13}$$

1.3.1 Top-Down Approach

The main point is that in all of these cases we have a clear statement of the two sides of the duality. On the field theory side we can write down a Lagrangian, which can be studied for example in perturbation theory at weak coupling, while the dual gravity theory is well understood in the supergravity limit. Furthermore, it is known in principle how to compute corrections in $1/N$ and the 't Hooft coupling λ (or its equivalent). We may therefore ask questions about the detailed dynamics of these theories, such as

> *"What is the shear viscosity η* of strongly coupled
> $\mathcal{N} = 4$ SUSY SU(N) *Yang-Mills theory in d=4?"*

or questions about their phase structure, such as

> *"Does ABJM theory have a superconducting ground state?"*

This is called the 'top-down' approach to applied AdS/CFT: we start out with a well-defined pair of dual theories, and then ask the type of questions a condensed-matter theorist, or a person with an interest in QCD, would ask about strongly coupled systems and answer it for the theories with well-defined holographic duals. Either way we compute answers to quantities at ultra strong coupling which are not known from any other approach. The advantage is that we can precisely identify the operators on each side, and rely on the established holographic dictionary. But a perhaps deeper point is that we know that these theories are embedded in a UV complete theory (string theory), which allows at least in principle to compute the quantum corrections in $1/N$ and λ in all regimes. This means, in particular, that the infinite N solution is a well-defined zeroth order approximation which can be systematically improved order by order, even though in practice this is often a tedious and painful task. It is also historically how the subject of applied holography started. Let us thus briefly[3] consider the question about the shear viscosity of $\mathcal{N}=4$ SYM in four dimensions. We will find an answer and a big surprise.

A First Look at Shear Viscosity

Viscosity is related to momentum transport and hence to the spatial components of the stress tensor. We begin by stating the definition of the shear viscosity in terms of a Kubo formula:

$$\eta = \lim_{\omega\to 0} \frac{1}{2\omega} \int \mathrm{dt}\,\mathrm{d\tilde{x}} e^{ik_\mu x^\mu} \Theta(t) \left\langle [T_{xy}(t,\vec{x}), T_{xy}(0,0)] \right\rangle \,. \tag{1.14}$$

This is a retarded two-point correlation function of the stress tensor of $\mathcal{N}=4$ SYM. According to the holographic dictionary the stress tensor is represented in the dual gravity theory by the metric and its fluctuations. Evidently what is called for is a determination of the two-point correlation function of stress tensor fluctuation in the dual $\mathcal{N}=8$ supergravity theory. For this we need to study fluctuations around the non-extremal (i.e. finite temperature) D3-brane solution,

$$ds^2 = H^{-1/2}(r)\left[-f(r)dt^2 + d\vec{x}^2\right] + H^{1/2}(r)\left[\frac{dr^2}{f} + r^2 d\Omega_5^2\right] \,. \tag{1.15}$$

with

$$H(r) = 1 + \ell^4/r^4, \qquad f(r) = 1 - r_0^4/r^4,$$

in the near-horizon limit. In particular we are looking for modes of the transverse and traceless part of the graviton fluctuation h_x^y. This allows us to determine the correlation function (1.14) and through it the shear viscosity. The answer is

$$\eta = \frac{\pi}{8} N^2 T^3 \,. \tag{1.16}$$

If we divide this quantity by the entropy density, we obtain an N-independent quantity, namely

$$\frac{\eta}{s} = \frac{1}{4\pi}\frac{\hbar}{k_B} \approx 6.08 \times 10^{-13} K s \,. \tag{1.17}$$

[3]We will go through this calculation in full detail later on in these lectures.

This quantity can in fact be measured for the gauge theory with the highest N known to play a role in nature, the SU(3) Yang-Mills theory with fundamental fermions, otherwise known as QCD. The relevant measurement has to be carried out in its quark gluon plasma phase (QGP), created at the relativistic heavy ion collider (RHIC) at Brookhaven and here at CERN by colliding heavy ions. The big surprise is that the measured result is numerically close to the answer we just calculated in $\mathcal{N} = 4$ SYM.

$$\frac{\eta}{s} \approx \mathcal{O}(1)\frac{1}{4\pi}\frac{\hbar}{k_B}Ks \qquad \text{QCD answer} \tag{1.18}$$

Let us summarize what we just found.

1. The holographic answer is very close to experimental value for $SU(3)$ YM with fundamental fermions in the plasma phase (QGP); and very different from perturbative predictions of this quantity

2. The calculation boiled down to solving the perturbations of the metric around a five dimensional black brane (planar black hole).
   The way we set up the calculation we never made reference to the full ten-dimensional geometry. This is possible because all other ten-dimensional fields $= 0$ or implied by uplift to $10D$ IIB supergravity, i.e. determined by pure gravity in five dimensions. It is thus within the consistent truncation down to the $SU(4)$ gauged supergravity we already mentioned above.

3. The result can be shown to be *universal* for pure gravity as it encapsulates a universal property of BH horizons (cf black hole membrane paradigm). That is to say that any holographic dual gravity whose EFT contains the Einstein-Hilbert term (see Eq. (1.3)) and certain symmetry requirements that allow us to define η in the first place, will have the same ratio η/s. This implies in particular that we would get the same answer for putative exact dual of QCD.

These points form the germ of what is commonly referred to as the bottom-up approach to applied AdS/CFT. In this approach we take at face value the fact that the theories for which precise duality relations are established do not share the same microscopics as our systems of potential interest. Nevertheless it is possible to establish results that are universal across all suitably defined (putative) examples of holography, so they should also apply to the putative duals of theories that do share (to the extent this is possible) the microscopics of realistic models. The computation of shear viscosity above serves as an illuminating case study: In the end it turned out that it was inconsequential that we computed it in the exact $\mathcal{N} = 4$ SYM theory, utilizing the exact supergravity approximation to IIB string theory. What mattered was that the terms contributing to the calculation were all within the pure gravity sector[4],

$$S = \frac{1}{2\kappa_{10}} \int d^{10}x\sqrt{-g}R + \cdots \tag{1.19}$$

which enjoys a great deal of universality due to the constrained nature of general relativity. Furthermore, the precise question we asked also dictated the background and its symmetries, namely the planar black hole in AdS_5. For example we needed to assume planar symmetry along the horizon to be able to define the momentum correlators (1.14) that determine the shear viscosity. Furthermore we were interested in the computation at finite temperature where hydrodynamics is well defined.

Once we have assembled all these ingredients, we could to all intents and purposes abstract the calculation away from the $\mathcal{N} = 4$ setting and think of a Lagrangian of the form (1.19), or more generally Eq. (1.3), as an effective AdS dual description of a large class of strongly-coupled field theories. We can take these ideas to their logical completion by freeing our computations away more generally from precisely established dual pairs, and instead build up effective AdS Lagrangians that suit our needs according to the questions we ask. This is known as the bottom-up approach to applied holography. A few points should be kept in mind. Ultimately it is important that such a low-energy theory can be embedded in a UV complete setting, or else the entire tool of holographic duality cannot be faithfully applied. Typically one appeals to the existence of a large landscape of AdS compactifications which contain somewhere a theory which includes the sector one is interested in. In fact, in well-known cases it has been possible, a posteriori, to embed a bottom up calculation in a specific UV-complete theory. An important technical point is

[4]The non-extremal D3 brane is supported by a five-form flux in the Ramond-Ramond sector of type IIB. However, from the effective five-dimensional point of view this simply contributes to the cosmological constant. Furthermore it does not play a role in the fluctuation analysis.

that it is often not clear in the bottom-up approach how to systematically improve the answer obtained within classical 'effective' AdS gravity beyond leading order, although sometimes semi-classical methods can be used for $1/N$ effects and argued to be universal. Note that this is not in contradiction to our earlier discussion of EFT: sure, any such Lagrangian would make sense up to some strong coupling scale $E_{\rm strong}$, but the duality in the first place relies on a UV completion for its mere existence. Whether a given EFT can be UV completed is an entirely different question, and one we cannot take for granted in the bottom-up approach. Note that in the top-down approach we start with the UV complete formulation from the outset, so the question never arises. After this cautionary remark let us summarize what we mean by the bottom-up approach to AdS/CFT.

1.3.2 Bottom-Up Approach

The bottom-up approach we just argued for, consists of the following procedure.

1. Identify the *minimal* AdS gravity model with the desired features (operator content, symmetries).

2. Determine the correct background solution (black brane, domain wall, ...) by solving the equations of motion subject to symmetry requirements etc.

3. Perform dual calculations, quite often of correlation functions, and infer physics of strongly coupled system. Of course plenty of results are known or can be obtained that go beyond correlation functions, for example in the area of non-linear response.

Let us underline once more the pros and cons of such an approach.

Pros

– More general applicability than top-down, yet easier to carry out

– Can lead to universal results (but must supply arguments, c.f. shear viscosity)

– Often gives new conceptual insights into the physics that can be abstracted and used in other contexts (e.g. non-equilibrium physics, conductivities,...)

Arguably it is the last of these three points which is the most exciting. In some sense one should view applied AdS/CFT as a powerful torch that allows us to shine light into an otherwise completely dark room and establish some idea of what is going on inside. Not many other tools exist that allow us to make significant progress in the realm of strongly coupled quantum field theory. This is particularly true when considering questions that concern far-from equilibrium physics and therefore intrinsically Lorentzian quantities. However, as alluded to above there are also some issues related to the bottom-up approach, which we now summarize.

Cons

– The rules not always 100% clear $\rightarrow$ important to develop effective holographic field theory more systematically.

– Really must *postulate* embedding into a consistent UV-complete quantum gravity theory except in a few happy cases, where the embedding is clear.

– Because of this, sometimes no systematic way to improve the infinite N answers is available, although EFT often allows to compute corrections in terms of a few free parameters.

In these lectures we will make use of both approaches but for the sake of pedagogy we shall explain the tools and results we focus on mostly on the basis of the bottom-up approach.

We will conclude this introductory lecture by asking the question what kind of physical systems we will mostly be interested in, noting that we already saw application to QCD in its plasma phase, which has given rise to much interesting activity in holography. One of the main other areas of physics, where holography has been applied fruitfully are the so-called quantum critical points in condensed-matter systems.

1.4 Quantum Critical Points

Quantum critical points (QCP) are locations of second-order phase transitions occurring at zero temperature[5]. We are all familiar with the notion of a second-order phase transition driven by thermal fluctuations. The theory that characterizes the critical phenomena appearing at such points in the phase diagram furnishes a classic application of the renormalization group and provides a beautiful example of how high-energy theory and condensed matter theory may advance hand in hand. In essence the critical thermal fluctuations at a phase transition point happen at all scales, which necessitates an RG analysis.

So how can a second-order phase transition happen at zero temperature where no thermal fluctuations occur? The answer is that a quantum critical transition occurs at loci where the ground state ($|0\rangle_g$) of a Hamiltonian, $H(g)$, depending on the continuous coupling parameter g is non-analytic at some critical coupling g_c. This usually happens at level-crossing points, where an energy gap Δ continuously goes to zero. Concurrent with this vanishing gap, we observe critical fluctuations in ξ, viz.

$$\begin{aligned} \Delta &\sim \xi^{-z} && \text{gap closes} \\ \xi^{-1} &\sim |g - g_c|^{\nu} && \text{correlation length} \end{aligned}$$

This implies that the critical theory, i.e. the continuum field theory that describes the phenomena occurring at the transition point, has a scaling symmetry

$$\begin{aligned} x &\mapsto \lambda x \\ t &\mapsto \lambda^z t\,. \end{aligned} \tag{1.20}$$

This is in general a scaling symmetry with asymmetric weights between space and time. In the literature it is often called a Lifshitz symmetry and a theory mani-

[5]The modern theory of quantum phase transitions is described in [4]. For a shorter introduction with emphasis on holography, see [5] and [6].

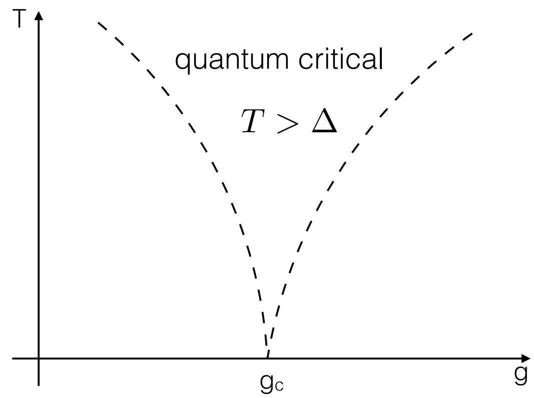


Figure 1.1: A typical phase diagram of a system with a quantum critical point at a coupling g_c. The quantum critical wedge, defined by the condition $T > \Delta$ is the region inside the dotted lines. In this region the physics is well described by a finite temperature deformation of the CFT emerging at the quantum critical point.

festing this symmetry a Lifshitz invariant field theory. Holographic duals for such cases are known, and we shall later have occasion to return to them. The astute reader will already have noticed that the case $z = 1$ looks awfully familiar. It is nothing but the scale symmetry of a Lorentz-invariant field theory. It is strongly believed that a local, relativistically invariant field theory exhibiting a scale symmetry of the form (1.20) is automatically invariant under the full conformal group. This is established as a theorem in two dimensions and strongly believed to be true in four dimensions. The scale invariance of the theory implies also that, for example, two point functions have a scale symmetry

$$\langle \phi(x)\phi(y)\rangle \to \lambda^{2\Delta}\langle \phi(\lambda x)\phi(\lambda y)\rangle$$

and similar for higher-order correlation functions. That is to say that quantum critical theories exhibit the same kinematics as AdS/CFT correlators.

As indicated above a whole region of the phase diagram can be described by the finite-T deformation of the quantum critical theory. This region, somewhat unintuitively, gets bigger as the temperature increases (see Fig. 1.1). Of course this behavior does not go on for an arbitrary range of temperature, as any realistic theory will cut this off at its UV (lattice) scale at the latest, but often much earlier. In AdS/CFT, if we can characterize the QCP itself, the finite-temperature deformation is easily obtained, with calculations often being easier than at zero temperature, even in Lorentzian signature. This ability to easily study finite-T versions of quantum critical points has established AdS/CFT as a powerful tool to study finite temperature physics of QGP in recent years. We conclude this chapter by briefly reviewing the canonical example of a many-body system exhibiting a $z = 1$ quantum critical point, the well-known Bose-Hubbard model.

1.4.1 The Bose Hubbard Model

The Bose Hubbard model is a lattice model of bosons, described by creation and annihiliation operators

$$\left[b_i, b_j^\dagger\right] = \delta_{ij} \qquad \left[b_i^\dagger, b_j^\dagger\right] = [b_i, b_j] = 0\,. \tag{1.21}$$

The Hamiltonian reads

$$H_{BH} = -t\sum_{\langle ij\rangle}(b_i^\dagger b_j + b_j^\dagger b_i) + U\sum_i n_i(n_i - 1) \tag{1.22}$$

where $n_i = b_i^\dagger b_i$ is the occupation number on site i and one has the $U(1)$ symmetry $b_i \to e^{i\phi} b_i$. The sum $\langle ij\rangle$ extends over nearest-neighbor pairs. Let us imagine, for the time being a two-dimensional lattice, but it is clear that other dimensions are also of interest and described by the same kind of Hamiltonian. One can study this theory in the canonical ensemble, fixing the total number of bosons, or in the grand canonical ensemble by adding a chemical potential term to (1.22). Either way the meaning of the terms multiplying the coupling t is to allow hopping of bosons across neighboring sites, while the U term suppresses occupations by more than one boson on each site. It is accordingly referred to as the on-site repulsion term. Different values of the dimensionless coupling $g \equiv U/t$ correspond to different phases of the system. When the coupling is very strong it is the on-site repulsion that wins and prevents bosons from moving from site to site. The system freezes into an insulator. When the coupling is small the hopping terms dominate and bosons are able to roam freely across the lattice. A detailed analysis shows that this corresponds to a superfluid phase. By continuity, somewhere in between the

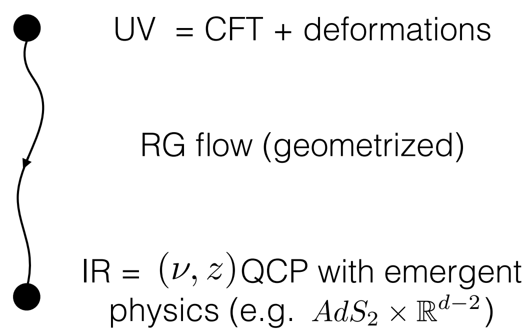


Figure 1.2: A quantum critical point can arise in the IR as the end point of an RG flow triggered from UV fixed point by adding a relevant deformation.

two must be a phase transition:

$$\begin{aligned} g \gg 1 &\qquad \text{bosons stuck} \to \text{insulator} \\ g \ll 1 &\qquad \text{hopping dominates} \to \text{superfluid} \\ g_c \sim O(1) &\qquad \text{superfluid-insultator (SI) transition} \, . \end{aligned}$$

A more detailed analysis reveals that this transition is indeed continuous, i.e. of second order, and defines a quantum critical point described by the so-called Wilson-Fisher (WF) fixed point theory, which is strongly coupled but can be controlled within an ϵ expansion. For more details see [4, 5].

Comments

1. As we shall see, AdS/CFT duals to CM systems do not need to be directly related to QCP in the UV. More commonly we study systems that flow to an interesting QCP in the IR, as illustrated in Fig. 1.2.

2. Vector-like (e.g. $O(N)$) theories have mean-field ($N \to \infty$) limit where one finds a QFT saddle point. Matrix-like (e.g. $\mathcal{N} = 4$ $SU(N)$ SYM) theories have large (λ, N) saddle points that solve *gravity* equations. The recently proposed case of SYK, as is understood currently, is somewhere in the middle.

Bibliography

[1] J. F. Donoghue, "General Relativity as an Effective Field Theory: the Leading Quantum Corrections," Phys. Rev. D **50** (1994) 3874 doi:10.1103/PhysRevD.50.3874 [gr-qc/9405057].

[2] T. Hartman, "Lectures on Quantum Gravity and Black Holes," http://www.hartmanhep.net/topics2015/

[3] J. M. Maldacena, "The Large N Limit of Superconformal Field Theories and Supergravity," Int. J. Theor. Phys. **38** (1999) 1113 [Adv. Theor. Math. Phys. **2** (1998) 231] doi:10.1023/A:1026654312961 [hep-th/9711200].

[4] S. Sachdev, "Quantum phase transitions," Wiley Online Library, 2007.

[5] S. A. Hartnoll, "Lectures on Holographic Methods for Condensed Matter Physics," Class. Quant. Grav. **26** (2009) 224002 doi:10.1088/0264-9381/26/22/224002 [arXiv:0903.3246 [hep-th]].

[6] C. P. Herzog, "Lectures on Holographic Superfluidity and Superconductivity," J. Phys. A **42** (2009) 343001 doi:10.1088/1751-8113/42/34/343001 [arXiv:0904.1975 [hep-th]].

Chapter 2

Pure States & Ensembles

2.1 Zero Temperature Backgrounds

After the introductory chapter we shall now turn to describing the technical details of the subject in order to equip the reader gradually with the knowledge to carry out concrete calculations. As mentioned above, we will proceed both using the top-down as well as the bottom-up point of view. From the former perspective valid backgrounds for holographic systems arise as so-called decoupling limits of brane constructions in string theory[1]. We have many examples, so let us just record a few ways in which different-dimensional AdS_{d+1} backgrounds arise as decoupling (near-horizon) limits of branes:

fieldtheory	$\longleftrightarrow$	gravity background
M5 branes (2,0) theory		$\mathrm{AdS}_7 \times \mathrm{M}^4$
D3 branes $\mathcal{N}$=4 SYM		$\mathrm{AdS}_5 \times \mathrm{M}^5$
M2 branes ABJM theory		$\mathrm{AdS}_4 \times \mathrm{M}^7$
(D1D5P) branes (4,4) SUSY		$\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{M}^4$

[1]Holography has arisen in the context of string theory and still many well-controlled examples arise from this route. It should be noted, however, that as far as we know the constructions described in this section are *not* the only ways in which AdS/CFT dual pairs can arise. In general any quantum gravity theory on an asymptotically anti-de Sitter space defines some conformal field theory, using the map between correlation functions to be described later.

Here $M^{4,5,7}$ are four-, five and seven dimensional manifolds which need to carry certain special geometrical structures in order to solve the constraints arising from the supergravity equations in the relevant dimension. There is a large literature on these and more general examples, which the interested reader may consult. For now we will confine our attention to the AdS_5, that is the D3-brane case.

2.1.1 Top Down Example: N $D3$ Branes in Type IIB Supergravity

We will motivate the correspondence between $\mathcal{N} = 4$ SYM and type IIB string theory following along the original arguments of Maldacena [1] (see also the review [2]). In this process we compare the low energy limit of a configuration of N D3 branes in perturbative string theory to that within the classical approximation, namely type IIB supergravity. We start with the latter.

The solution corresponding to the supergravity limit of a stack of N D3 branes in type IIB string theory takes the following form. We have a line element

$$ds^2 = \frac{1}{\sqrt{f}} \underbrace{\left(-dt^2 + \sum_{i=1}^{3}(dx^i)^2\right)}_{D3\ \text{world}-\text{volume}} + \sqrt{f}\, \underbrace{\left(dr^2 + r^2 d\Omega_5^2\right)}_{\text{transverse space}} \tag{2.1}$$

as well as the self-dual five form

$$F_{(5)} = (1+*)dt \wedge \prod_{i=1}^{3} dx^i \wedge df^{-1}, \tag{2.2}$$

with

$$f = 1 + \frac{\ell^4}{r^4}, \qquad \ell^4 = 4\pi g_s(\alpha')^2 N\,, \tag{2.3}$$

where $(\alpha')^2 = \ell_s^4$. The directions t, x^i form the world-volume of the D3 brane, i.e. the directions of space-time filled by the brane. The coordinate r is the radius of the remaining ('transverse') directions which contain a five sphere with round metric $d\Omega_5^2$. As indicated the parameter ℓ with dimensions of length is related to the number N of quanta of five-form flux,

$$\int_{S^5} F = N \,, \tag{2.4}$$

which is in turn given by the number of D3 branes sourcing the Ramond-Ramond five form field. We shall now take the so-called decoupling limit, that is we zoom in on the near-horizon region $r \ll \ell$. In this limit, the leading order expression for the line element reads

$$ds^2 = \underbrace{\frac{r^2}{\ell^2}\left(-dt^2 + \sum_{i=1}^{3}(dx^i)^2\right) + \frac{\ell^2 dr^2}{r^2}}_{AdS_5} + \underbrace{\ell^2 d\Omega_5^2}_{S^5} . \tag{2.5}$$

An excitation with energy E measured at infinity has energy E_p at a point p, located at radial coordinate position r_p, given by $E = E_p\sqrt{-g_{tt}}$, so that we find

$$E = f^{-1/4}E_p \sim \frac{r}{\ell}E_p \,. \tag{2.6}$$

This equation is at the basis holographic interpretation of the radial direction r as the energy scale of the field theory, with the changing geometry as a function of time corresponding to the flow of the field theory across scales. The more conceptual version of this statement is that the holographic dual geometrizes the RG flow of the field theory.

A finite energy excitation $E_p \neq 0$ will be seen to have lower and lower energy at the boundary as the point p is brought closer and closer to the horizon at $r = 0$. In the near-horizon limit we find effectively two decoupled sectors of low-energy excitations: firstly the massless type IIB supergravity excitations of the asymptotically flat region near infinity and secondly the arbitrary finite energy excitations of the near-horizon region, which due to the redshift effect we just described also appear massless to a boundary observer.

A similar decoupling happens from the perspective of the D3 brane world-volume theory, where the decoupling limit results in a sector comprising the $\mathcal{N} = 4$ SYM theory on the one hand, and asymptotic type IIB theory on the other. Maldacena then argued that the low energy limit gives two alternative perspectives on the same physics, namely

(a) $\Big[$*near horizon* IIB string theory $\Big] \bigoplus \Big[$asymptotic type IIB $\Big]$

(b) $\Big[\mathcal{N}=4$ SYM with gauge group $SU(N)$ $\Big] \bigoplus \Big[$asymptotic type IIB$\Big]$

Both a) and b) are complete descriptions of the system at low energies and both a) and b) result in two decoupled theories, one of which is asymptotic type IIB string theory (supergravity). We thus conclude that the other two theories must also be equivalent. That is string theory in the near-horizon limit of the D3 brane geometry must be equivalent to the $\mathcal{N}=4$ SYM theory with gauge group $SU(N)$. This striking observation is the simplest case of a holographic duality and we have essentially followed along a top-down line of arguments to arrive at this conclusion. Of course, even though we use concrete theories with given operator content and interactions, the above arguments are heuristic and should be viewed as merely motivating the conjecture of the duality. Proving it is an altogether different issue, but we alluded to this point already in the introduction.

Comments

Let us conclude this section by highlighting a few important points.

1. Calculations in (applied) holography start in the first instance by identifying the right gravitational background. This should be viewed as geometrizing the state $|\Psi\rangle$ of the dual field theory for a given problem

2. The ground state of the system should therefore correspond to a zero-temperature geometry. In the case of the $\mathcal{N}=4$ SYM theory in $d=4$, this ground state corresponds to the geometry $\mathrm{AdS}_5\times\mathrm{S}^5$. We introduced the idea that the extra 'holographic' direction of AdS_5, denoted here by r, should be identified with the energy scale of the field theory.

3. The $\mathcal{N}=4$ theory is a conformal field theory, so that its RG flow is trivial. Alternatively we could say that it arises as an RG fixed point, which we can deform by adding relevant operators to trigger non-trivial RG flows.

4. The five sphere S^5, with isometry group $SO(6)$, arises due to the internal SO(6) symmetry of the theory, while the AdS_{d+1} space with isometry group $SO(2,4)$ geometrically encodes the conformal symmetry. Indeed the conformal group in four dimensions is SO(2,4) and we can identify the action of each generator in terms of a Killing vector of AdS_5.

Interlude: Geometry and Isometries of AdS

We pause to fill in some details of the geometry of anti-de Sitter space and to identify its isometry group, which coincides with the conformal group. For this it is most convenient to work in the embedding formalism. Let us therefore start with the space $\mathbb{M}^{2,d}$, i.e. $d+2$ dimensional flat space with metric

$$ds^2 = -dX_0^2 - dX_{d+1}^2 + \sum dX_i^2 \,. \tag{2.7}$$

Anti-de Sitter space in $d+1$ dimensions is defined as the hypersurface

$$X_0^2 + X_{d+1}^2 - \sum X_i^2 = \ell^2 \,, \tag{2.8}$$

which defines the quadratic form $\hat{\eta} = \mathrm{diag}\left[- - + \cdots +\right]$. The embedding equation (2.8) can be solved in terms of $d+1$ independent coordinates, in several different ways. In each case the induced metric is a local form of the anti-de Sitter metric. We mention two frequently encountered examples

Global Coordinates

We can solve the embedding constraint by choosing

$$X_0 = \ell \cosh\rho \cos t \,, \quad X_{d+1} = \ell \cosh\rho \sin t \,, \quad X_i = \ell \sinh\rho \Omega_i \,, \tag{2.9}$$

where Ω_i are a set of angles parametrizing a unit $d-1$ sphere. The induced metric on the hyperboloid takes the form

$$ds^2 = \ell^2 \left(-\cosh\rho^2 dt^2 + d\rho^2 + \sinh\rho^2 d\Omega_{d-1}^2\right) \,. \tag{2.10}$$

For this metric, one usually takes the range of the coordinate $t \in (-\infty, \infty)$, i.e. one considers formally the covering space of the hyperboloid.

Poincaré Coordinates

We may also solve the embedding constraint by the choice

$$\begin{aligned} X_0 &= \frac{1}{2r}\left[1 + r^2\left(\ell^2 + x_i^2 - t^2\right)\right] \\ X_d &= \frac{1}{2r}\left[1 - r^2\left(\ell^2 - x_i^2 + t^2\right)\right] \\ X_i &= \ell r x_i \\ X_{d+1} &= \ell r t \end{aligned} \tag{2.11}$$

Now the induced metric on the hyperboloid is

$$ds^2 = \ell^2\left[\frac{dr^2}{r^2} + r^2\left(-dt^2 + dx_i^2\right)\right] . \tag{2.12}$$

The Poincaré embedding covers only half of the AdS hyperboloid, but it is often very useful to consider, as it corresponds to studying the boundary field theory simply on flat Minkowski space.

Isometries

In principle the isometries of a given spacetime can be found by solving the Killing vector equation, $\nabla_{(A}\xi_{B)} = 0$. But in the present case we can make use of the embedding formalism to find the isometries in more elementary way. Evidently the symmetries of AdS_{d+1} are those transformations on the embedding coordinates which leave the embedding equation (2.8) invariant. Thus, writing

$$X^T\hat{\eta}X = X^T R^T \hat{\eta} R X \qquad \Rightarrow \qquad R^T\hat{\eta}R = \hat{\eta}\,. \tag{2.13}$$

Here R is a linear transformation acting on the vector of embedding coordinates X. Evidently it must leave the quadratic form $\hat{\eta}$ invariant, so that we see immediately that the isometry group is $SO(2,d)$. This is nothing but the conformal group in d

dimensions. Writing the linear transformation

$$R = \exp\left(i\alpha^A T_A\right) \tag{2.14}$$

and working to infinitesimal order in α^A we find that the group is generated by matrices T_A, such that $T_A^T = -\hat{\eta} T_A \hat{\eta}$, i.e. matrices that are antisymmetric with respect to the quadratic form $\hat{\eta}$. The infinitesimal generators of this group, acting on $\mathbb{M}^{2,d}$, are then given either by boosts, such as

$$\xi_b = X_0 \frac{\partial}{\partial X_1} + X_1 \frac{\partial}{\partial X_0} \tag{2.15}$$

or by rotations, such as

$$\xi_r = X_1 \frac{\partial}{\partial X_2} - X_2 \frac{\partial}{\partial X_1} \,. \tag{2.16}$$

Now all we need to do is project the action of these generators onto the hyperboloid. This is very simple: Let us denote by y^μ the set of $d+1$ independent coordinates solving the constraint (2.8) in a given coordinate system. We then have

$$\frac{\partial}{\partial X^A} = \frac{\partial y^\mu}{\partial X^A} \frac{\partial}{\partial y^\mu} \,. \tag{2.17}$$

The projection matrix $\frac{\partial y^\mu}{\partial X^A}$ is easiest calculated by defining it as the pseudoinverse of $\frac{\partial X^A}{\partial y^\mu}$, which itself can be directly calculated from the explicit solution of the embedding constraint. Let us recall that the generators can be arranged into an anti-symmetric matrix,

$$J_{M,N} = \begin{pmatrix} J_{\mu\nu} & J_{\mu,d+1} & J_{\mu,d+2} \\ -J_{\nu,d+1} & 0 & D \\ -J_{\nu,d+2} & -D & 0 \end{pmatrix} . \tag{2.18}$$

Here $J_{\mu\nu}$ are the usual generators of the Poincaré group in d dimensions, while D is the dilatation generator and

$$J_{\mu,d+1} = \tfrac{1}{2}\left(P_\mu - K_\mu\right) \qquad J_{\mu,d+2} = \tfrac{1}{2}\left(P_\mu + K_\mu\right) \tag{2.19}$$

with P_μ the generators of translation and K_μ the special conformal generators. We thus learn that the conformal group in d dimensions is generated by the set

$$\{J_{\mu\nu}, P_\mu, K_\mu, D\} \ . \tag{2.20}$$

These generators satisfy the algebra of $SO(2,d)$, which becomes the conformal algebra, when rewritten in terms of the set (2.20).

$$\begin{aligned} [J_{\mu\nu}, K_\rho] &= -i\left(\eta_{\mu\rho}K_\nu + \eta_{\nu\rho}K_\mu\right) \,, \qquad [J_{\mu\nu}, D] = 0\,, \qquad [D, K_\mu] = iK_\mu \\ [D, P_\mu] &= -iP_\mu \qquad [P_\mu, K_\nu] = 2iJ_{\mu\nu} - 2\eta_{\mu\nu}D\,. \end{aligned} \tag{2.21}$$

while the rest satisfies the usual Poincaré commutation relations

$$[J_{\mu\nu}, J_{\rho\sigma}] = -i\eta_{\mu\rho}J_{\nu\sigma} + \text{perm.} \qquad [J_{\mu\nu}, P_\rho] = -i\left(\eta_{\mu\rho}P_\nu + \eta_{\nu\rho}P_\mu\right) \,. \tag{2.22}$$

We will go illustrate the abstract discussion above by going explicitly through the case of AdS_3.

The Example of AdS$_3$

In AdS$_3$ the quadratic form $\hat{\eta}$ is symmetric, in the sense that $\hat{\eta} = \text{diag}[++--]$. This is reflected in the fact that the conformal group $SO(2,2) \cong SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ factors. This is not the case for $SO(2,d)$ for $d > 2$. We will see this more explicitly once we have found the algebra. We solve the constraint by

$$X_0 = \ell\cosh\rho\cos t\,, \quad X_1 = \ell\sinh\rho\sin\phi\,, \quad X_2 = \ell\sinh\rho\cos\phi\,, \quad X_3 = \ell\cosh\rho\cos t\,, \tag{2.23}$$

which induces the metric

$$ds^2 = \ell^2\left(-\cosh\rho^2 dt^2 + d\rho^2 + \sinh\rho^2 d\phi^2\right) \,, \tag{2.24}$$

known as the global metric on AdS$_3$. By computing the projection matrix onto the hyperboloid, it is straightforward to establish that the two Killing vector already mentioned above take the form

$$\begin{aligned}
\xi_b &= \cos t \sin\phi \partial_\rho - \sin t \sin\phi \tanh\rho \partial_t + \cos t \cos\phi \coth\rho \partial_\phi \\
\xi_r &= \partial_\phi \,.
\end{aligned} \tag{2.25}$$

It is immediately obvious that the second of these is a Killing vector of the metric (2.24). Similarly, the first can be shown to be so as well, after a small calculation. In fact all the Killing vectors of AdS_3 can be constructed in this way. We now quote them, in a convenient basis

$$\begin{aligned}
\zeta_{-1} &= \tfrac{1}{2}\left[\tanh\rho e^{-i(t+\phi)}\partial_t + \coth\rho e^{-i(t+\phi)}\partial_\phi + i e^{i(t+\phi)}\partial_\rho\right] \\
\zeta_0 &= \tfrac{1}{2}\left(\partial_t + \partial_\phi\right) \\
\zeta_1 &= \tfrac{1}{2}\left[\tanh\rho e^{i(t+\phi)}\partial_t + \coth\rho e^{i(t+\phi)}\partial_\phi - i e^{i(t+\phi)}\partial_\rho\right] \,, \qquad (2.26)
\end{aligned} \tag{2.27}$$

with a similar set of expressions for $\bar{\zeta}_{\pm 1}$ and $\bar{\zeta}_0$. One can now verify that they satisfy two decoupled $\mathrm{SL}(2,\mathbb{R})$ algebras under Lie brackets:

$$[\zeta_1, \zeta_{-1}] = 2\zeta_0 \qquad [\zeta_1, \zeta_0] = \zeta_1 \qquad [\zeta_{-1}, \zeta_0] = -\zeta_{-1} \,, \tag{2.28}$$

where the bracket between two Killing vectors is defined in terms of their Lie bracket

$$[\zeta_i, \zeta_j] := i\left\{\zeta_i, \zeta_j\right\}_{\mathrm{Lie}} \,. \tag{2.29}$$

This is to be compared with the standard $\mathrm{SL}(2,\mathbb{R})$ algebra, in terms of its generators L_i.

$$[L_1, L_{-1}] = 2L_0 \qquad [L_1, L_0] = L_1 \qquad [L_{-1}, L_0] = -L_{-1} \tag{2.30}$$

The second $\mathrm{SL}(2,\mathbb{R})$ algebra is contained in the barred generators $\bar{\zeta}_i$. We can write the conformal algebra of $SO(2,2)$ in the more conventional form, using the following relations

$$\begin{aligned}
D &= i\left(L_0 + \overline{L}_0\right) \qquad K_0 = i\left(L_{-1} + \bar{L}_{-1}\right) \qquad K_1 = i\left(L_{-1} - \bar{L}_{-1}\right) \\
J_0 &= i\left(L_1 + \bar{L}_1\right) \qquad J_1 = L_1 - \bar{L}_1 \qquad J_{01} = i\left(\bar{L}_{-1} - L_{-1}\right) \,.
\end{aligned} \tag{2.31}$$

We shall now abstract these features from the concrete top-down example that exhibits them and will promote them to general principles or lessons to guide us in the construction of bottom-up backgrounds.

2.1.2 Bottom-Up Example: AdS$_{d+1}$ as an RG Fixed Point

However, with hindsight there is a more conceptual way to arrive at the geometry of AdS$_{d+1}$, which furthermore does not make any reference to the string-theory ingredients we needed to appeal to in the previous section. We start by restating the well-known fact that a CFT can be seen as arising as an RG fixed point of a Lorentz invariant field theory, characterized as a point where the beta function vanishes

$$\mu\partial_\mu g = \beta(g(\mu)) = 0. \tag{2.32}$$

We now take the point of view that, in order to holographically encode this fixed point, we would like to construct a geometry, involving one more spatial dimension, which encodes the resulting scale-invariance as an isometry. In an isotropic system, scale symmetry acts on the coordinates as

$$x^\mu \to \lambda x^\mu\,, \qquad r \to \lambda^{-1} r \tag{2.33}$$

The latter transformation simply reflects the identification of the holographic direction r as the energy scale. We will now find most general metric that is invariant under (2.33) and which exhibits Poincar invariance in (t, x^i). We therefore start with the ansatz

$$ds^2 = g(r)\eta_{\mu\nu}dx^\mu dx^\nu + h(r)dr^2 \tag{2.34}$$

for two functions of the energy scale r and where $\eta_{\mu\nu}$ is the Minkowski metric in d dimensions. This of course is dictated by the Poincaré symmetry. Under the scale transformation

$$ds^2 \to g(r/\lambda)\lambda^2\eta_{\mu\nu}dx^\mu dx^\nu + \frac{h(r/\lambda)}{\lambda^2}dr^2. \tag{2.35}$$

Thus we must have that g and h are homogenous functions of degree two

$$g(r/\lambda) = \frac{1}{\lambda^2} g(r) \qquad \Longrightarrow \qquad g(r) = \frac{r^2}{c_1^2}$$

$$h(r/\lambda) = \lambda^2 h(r) \qquad \Longrightarrow \qquad h(r) = \frac{c_2^2}{r^2}$$

for two undetermined parameters c_i. In summary we thus have

$$ds^2 = \left(\frac{r}{c_1}\right)^2 \eta_{\mu\nu} dx^\mu dx^\nu + \frac{dr^2}{r^2} c_2^2 \tag{2.36}$$

We can absorb one of the coefficients via a rescaling $r \to \frac{c1}{c2} r$ and then identify $c_2 = \ell$ to obtain the metric of AdS_{d+1} of radius ℓ.

$$ds^2 = \left(\frac{r}{\ell}\right)^2 \eta_{\mu\nu} dx^\mu dx^\nu + \frac{\ell^2}{r^2} dr^2 \tag{2.37}$$

One often sees a different coordinate form of this metric, where we use the inverse of the radial coordinate

$$z := \frac{\ell^2}{r} \tag{2.38}$$

The resulting metric takes the form

$$ds^2 = \left(\frac{\ell}{z}\right)^2 \left[\eta_{\mu\nu} dx^\mu dx^\nu + dz^2\right] \tag{2.39}$$

often referred to as the 'Poincaré metric', or more accurately 'Poincaré patch' of anti-de Sitter space. This presentation of AdS_{d+1} makes it clear that it is conformally flat, which is often quoted in the holographic literature by saying that in this coordinate representation of AdS_{d+1} one studies the dual field theory on flat Minkowski space. Evidently we have arrived at the conclusion that the metric of AdS_{d+1} describes a conformal field theory in a much simpler manner than before. But there is also a big part of the earlier story that we have not reconstructed. Namely the dynamics that gives rise to this metric, as well as the fluctuations around it. The way that this enters the bottom-up story is that we simply seek the simplest diffeomorphism invariant gravitational action that admits the metric (2.39) as a solution. Combined with the gravity as an EFT reasoning from Chapter 1, this leads us – at lowest order – to the Einstein-Hilbert action with negative cosmological constant

$$S = \frac{1}{2\kappa^2} \int d^{d+1}x \sqrt{-g} \left(R + \frac{d(d-1)}{\ell^2} \right) \tag{2.40}$$

The present RG point of view also suggests the form of the metric for situations with broken scale invariance, while preserving translational and rotational invariance in the x^i directions. We consider deformed metrics of the form

$$ds^2 = \frac{R^2}{z^2} \left(-f(z)dt^2 + g(z)dz^2 + h(z) \sum_i (dx^i)^2 \right) \tag{2.41}$$

Diffeomorphism invariance $z \to z'(z)$ means that, without loss of generality, we can set $g = 1$. If the RG flow preserves Lorentz invariance, $f(z) = h(z)$. In order to recover the RG fixed point, i.e. the geometry of AdS_{d+1}, in UV, we must require that

$$\lim_{z \to 0} f(z) = c \tag{2.42}$$

for some constant c. We now consider a very important case of such a deformation: a finite temperature T. We could now immediately adopt the bottom-up perspective which would be to insert the ansatz (2.41) into the equations of motion of (2.40) to find the most general solution involving a non-trivial deformation of the form (2.41). We would be led to the planar black hole solution as the answer. But let us once more see how this answer actually arises from the top-down perspective, which will confirm our intuition leading to the metric ansatz as well as the use of the Einstein-Hilbert action as an effective theory.

2.2 Finite Temperature Backgrounds

In order to add some variety to our arsenal of models, let us consider, instead of the D3 brane case in ten dimensions the case of the M2 brane in eleven dimensions. This means that instead of finding the finite-temperature generalization of the above $\mathrm{AdS}_5 \times \mathrm{M}^5$ type solution, we will instead perform that exercise for the

$AdS_4\times$ M^7 type family of backgrounds. Hence we expect to find a duality relation involving AdS_4 as the non-compact geometry. So what is the brane configuration we should consider? The answer is provided by the so-called non-extremal M2 brane, whose worldvolume theory is a $2+1$ dimensional field theory with a $3+1$ dimensional dual geometry. The fact that non-extremal solutions correspond to the finite temperature ensemble, while extremal ones to zero temperature is something we will see in more detail below, but for now we accept it as a given and proceed with the analysis.

2.2.1 Non-Extremal M2 Brane

Let us recall that the M2 brane is described a low energies by a $2+1$ dimensional QFT and this is expected to be holographically dual to a theory living on a spacetime that is asymptotically $AdS_4\times$ M^7 for some suitable seven manifold M^7. In the supergravity approximation the relevant part of the eleven dimensional theory follows from the action

$$S_{11D} = \frac{1}{2\kappa_{11}^2}\int d^{11}x\sqrt{-g}R - \frac{1}{4\kappa_{11}^2}\int\left(F_{(4)}\wedge\star F_{(4)} + \frac{1}{3}A_3\wedge F_{(4)}\wedge F_{(4)}\right). \quad (2.43)$$

One can show that there exists a solution with the following metric

$$ds^2 = H(r)^{-2/3}\left[-f(r)dt^2 + \sum_{i=1}^{2}(dx^i)^2\right] + H(r)^{1/3}\left[\frac{dr^2}{f(r)} + r^2 d\Omega_7^2\right] \quad (2.44)$$

, where

$$H(r) = 1 + \frac{\ell^6}{r^6} \quad (2.45)$$

$$f(r) = 1 - \frac{r_0^6}{r^6}$$

Finally, there is a non-trivial flux of the four form and the seven form fields

$$\star\, F_{(4)} = F_{(7)} = 6\ell^6\,\mathrm{vol}(\mathbb{S}^7) \quad (2.46)$$

This provides a solution of the eleven-dimensional field equations, known as the non-extremal M2-brane solution. In this case the map between the anti-de Sitter curvature, the eleven dimensional Planck mass and the number of local degrees of freedom (central charge) can be shown to be given by

$$\ell^9 \pi^5 = N^{3/2} \kappa_{11}^2 \tag{2.47}$$

The peculiar scaling of $N^{3/2}$ appears puzzling from a gauge-theory perspective, but can be understood for example via supersymmetric localization techniques [3]. In order to find the appropriate generalization of the ground state geometry AdS_{d+1} to finite temperature we should again take the near-horizon decoupling limit $r \ll \ell$. We find the line element

$$ds^2 = \underbrace{\frac{r_0^4}{u^2 \ell^4}\left[-f(u)dt^2 + d\vec{x}^2\right] + \frac{\ell^2}{4f(u)}\frac{du^2}{u^2}}_{\text{black hole in } AdS_4} + \ell^2 d\Omega_7^2, \tag{2.48}$$

where we have used the new dimensionless radial coordinate

$$u = r_0^2/r^2 \tag{2.49}$$

in terms of which $f = 1 - u^6$. As indicated in the formula, the four-dimensional part of the metric we find describes an uncharged black brane in AdS_4. This makes intuitive sense: we know from the classic works of Bekenstein and Gibbons and Hawking that black holes douse their environment in a thermal bath of excitations at the Hawking temperature T_H. Furthermore we can view the gravity partition function as naturally defining a grand canonical potential. In the context of holography, as we will now explore in detail, this fact about semi-classical gravity has the natural interpretation that the field theory on the boundary is now studied in a thermodynamic ensemble rather than in its (pure) ground state. For this purpose we will now turn to some basic notions of gravitational thermodynamics, along the lines of the classic work of Gibbons and Hawking [4].

2.2.2 Some Thermodynamics

There are several (mutually consistent) ways to see that black holes are thermodynamic objects with *temperature* T and *entropy* S. For example one can prove a set of results establishing a close analogy between the behavior of black hole horizons, such as their non-decreasing nature in time, and the laws of thermodynamics, such as the second law. This is essentially a Lorentzian perspective, where we see how the evolution of Einstein equations constrains the properties of black holes. A more concise derivation of equilibrium black-hole thermodynamics can be given using Euclidean techniques, and we will present this approach here. Let us formally consider a path integral for gravity in anti-de Sitter space

$$Z = \int \mathcal{D}g e^{iS[g,\cdots]} \xrightarrow[\text{for static solutions}]{\text{Euclideanize}} \int \mathcal{D}g e^{-S_E[g_E,\cdots]}\,, \tag{2.50}$$

where as emphasized in the equation, we take advantage of the static (more generally stationary) nature of the solution to define a Euclidean version of the path integral via Wick rotation. Notice that in principle we allow for dependence of the action on additional fields, which is often the case in holographic applications. We then look at the limit where the theory is semiclassical, i.e. for small Newton constant. In this case we may evaluate the Euclidean partition function via the saddle point method:

$$Z \overset{\text{saddle pt.}}{=} e^{-S_E[\bar{g}_e,\cdots]} \int \mathcal{D}\left[\delta\Phi\right] e^{-\frac{\delta^2 S}{\delta\Phi\delta\Phi}\delta\Phi\delta\Phi}\,. \tag{2.51}$$

where we have denoted all the fluctuations around the saddle point as $\delta\Phi$. There is a local gauge redundancy in the form of diffeomorphism invariance. As in Yang-Mills theory this can be dealt with using the Fadeev-Popov method, rendering the quadratic fluctuation integral well defined, so that one can make sense of this path integral to 1−loop around a stationary solution. We shall now argue that the Euclidean path integral is not well defined unless the Euclidean time direction is identified with a certain period $\tau \sim \tau + \beta$, to be determined. People familiar with the path-integral formulation of statistical physics will immediately recognize that this implies that the semi-classical partition function thus describes a thermodynamic potential

$$Z = e^{-\beta F[T,S]} \qquad \text{(free energy.)} \tag{2.52}$$

at temperature $1/\beta$. Let us now derive this temperature.

We euclideanize the metric by defining $t = -i\tau$, and for convenience we also rescale $\ell \to 2\ell$ and $u \to z = \alpha u$, with $\alpha = 4r_0^2/\ell^3$. The decoupling limit of the non-extremal M2 brane solution now has the line element

$$ds^2 = \frac{\ell^2}{z^2}\left[f(z)d\tau^2 + \frac{dz^2}{f(z)} + \sum_i (dx^i)^2\right] + \ell^2 d\Omega_7^2, \tag{2.53}$$

where $f(z) = 1 - u^3 = 1 - z^3/z_h^3$. For our purposes the interesting part of this metric is the $\tau - z$ plane, on which we now focus our attention. We expand near the horizon $f(z) = f_+'(z_h)(z - z_h)$ and write

$$\begin{aligned} ds^2 &= \ell^2 \left[\frac{f_+'}{z_+^2} d\tau^2 + \frac{dz^2}{z_h^2 f_+'(z - z_h)}\right] \\ &= \ell^2 \left[\rho d\left(\frac{f_+'}{2}\tau\right)^2 + d\rho^2\right] \\ &= \ell^2 \left[\rho^2 d\phi^2 + d\rho^2\right] . \end{aligned} \tag{2.54}$$

Here we have defined the new radial variable ρ via the differential equation

$$\frac{dz}{z_h\sqrt{f_+'(z - z_h)}} = d\rho\,, \tag{2.55}$$

which is easily solved to yield

$$\rho = \frac{2}{z_h\sqrt{f_+'}}\sqrt{z - z_h}\,. \tag{2.56}$$

As can be argued for example by invoking the Gauss-Bonnet theorem, every cone has a singular curvature localized at its tip. In the present context this singularity would render the semi-classical partition function ill defined. Thus we need to impose that the potential deficit angle of the cone vanishes by imposing $\phi \sim \phi + 2\pi$.

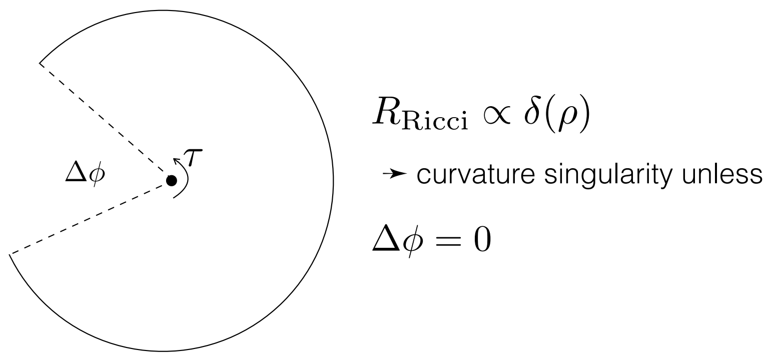


Figure 2.1: The $\tau - \rho$ plane of a static black hole is given by a cone with angular deficit $\Delta\phi$. Unless this deficit vanishes the Euclidean partition function is not well defined. Requiring $\Delta\phi = 0$ imposes a fixed periodicity on the Euclidean time coordinate and therefore a finite temperature. This temperature is the famous Hawking temperature.

This relation ensures that the metric (2.54) is nothing but the two-dimensional Euclidean plane written in polar coordinates, and thus has nothing singular about it. Rewritten in the original variables this yields the identification

$$\tau + \frac{4\pi}{f'_+} \equiv \tau + \beta. \tag{2.57}$$

We thus conclude that the Euclidean path integral defines a thermodynamic ensemble with a specific temperature

$$T_H = \frac{1}{\beta} = \frac{1}{4\pi} f'_+(z_h) \tag{2.58}$$

This is the famous Hawking temperature of a black hole. Notice that the periodic identification of the τ coordinate must hold throughout the (Euclidean) spacetime and in particular at the holographic boundary at $z = 0$. It follows immediately that the dual QFT is in a finite temperature ensemble at T_H. This is the result we presaged above.

The entropy of this thermal ensemble is a bit harder to obtain, as we must evaluate $S_E(\bar{g})$. We will do this in detail later, but for now let us quote the result. One finds the expression

$$S = -\frac{\partial F}{\partial T} = \frac{(4\pi)^d \ell^{d-1}}{16\pi G_N d^d} \mathrm{vol}\left(\mathbb{R}^{d-1}\right) T^{d-1} \tag{2.59}$$

$$= \frac{8\sqrt{2}\pi^2}{27} N^{3/2} T^2 \,, \tag{2.60}$$

where in the second line we have plugged in the parameters of the M2 brane solution for concreteness. Once more we see the peculiar $N^{3/2}$ scaling of the number of degrees of freedom contributing to the entropy. On can also show, via an independent calculation that this expression is equivalent to the familiar Bekenstein-Hawking result

$$S = \frac{A}{4G_N} \,, \tag{2.61}$$

where A is the area (density) of the black hole horizon, and thus the present derivation is seen to be reassuringly consistent with the original Lorentzian approach of Penrose, Hawking and others.

Bibliography

[1] J. M. Maldacena, "The Large N Limit of Superconformal Field Theories and Supergravity," Int. J. Theor. Phys. **38** (1999) 1113 [Adv. Theor. Math. Phys. **2** (1998) 231] doi:10.1023/A:1026654312961 [hep-th/9711200].

[2] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri and Y. Oz, "Large N Field Theories, String Theory and Gravity," Phys. Rept. **323** (2000) 183 doi:10.1016/S0370-1573(99)00083-6 [hep-th/9905111].

[3] N. Drukker, M. Mariño and P. Putrov, "From Weak to Strong Coupling in Abjm Theory," Commun. Math. Phys. **306** (2011) 511 doi:10.1007/s00220-011-1253-6 [arXiv:1007.3837 [hep-th]].

[4] G. W. Gibbons and S.W. Hawking, *Action integrals and partition functions in quantum gravity*, *Physical Review D* **15** (1977) pg. 2752-2756.

Chapter 3

Thermodynamics

3.1 Introduction to Holographic Renormalization

(See also [1, 2] and references therein.)
A word on indices: Whenever we refer to a ten or eleven dimensional spacetime which typically solves supergravity equations of motion we will denote its spacetime coordinates by capital Latin indices $M, N, \ldots$. This is mostly relevant in top-down discussions. We will always denote the coordinates of the d dimensional QFT dual to our AdS type solutions by Greek letters $\mu, \nu, \ldots$. If we only focus on the AdS_{d+1} part of the dual geometry, as is often the case in bottom-up discussions, we shall denote its coordinates with lower-case Latin indices $a, b, \ldots$. That is we have $a \in (z, \mu)$, or sometimes $a \in (r, \mu)$, etc depending on which coordinates we use to describe the AdS radial direction.

In the previous chapter we saw that the partition function of AdS_{d+1} gravity, in the semi-classical limit, defines a thermodynamic potential with temperature $T = \frac{1}{\beta}$. We calculated this temperature by fixing the period of Euclidean time to avoid a conical singularity at the locus where there would be a horizon in Lorentzian signature. We also quoted, without giving the details, the result for the free energy. We shall now see how to obtain it. Along the way we will have to deal with a divergent on-shell action in a manner quite analogous to the renormalization procedure in ordinary QFT[1]. We will now give a first treatment of this subject, postponing a much more systematic treatment to a later chapter.

When evaluating the thermodynamic free energy, one encounters the gravitational on-shell action,

$$Z \propto e^{-S_E}, \tag{3.1}$$

where S_E is the Euclidean action. This will turn out to be a divergent quantity. In order to see the issues involved, let us first try and evaluate this for the simplest case, namely the vacuum background AdS_{d+1}. We have the bulk action

$$S_{EH} = \frac{1}{16\pi G_N} \int_{\mathcal{M}} d^{d+1}x \sqrt{-g} \left(R + \frac{d(d-1)}{\ell^2} \right) , \tag{3.2}$$

into which we would like to substitute the classical background solution, that is evaluate its value on the saddle point. But before we do so we need to address an important subtlety: on manifolds with boundary, (3.2) does not give a well-defined variational principle. The Einstein-Hilbert term contains terms proportional to second derivatives, so we need to add a boundary term to cancel the terms arising from their variation. Were we not to add the boundary term, specifying Dirichlet or Neumann conditions (or mixed) at the boundary would not be sufficient to imply the equations of motion. Choosing to add the so-called York Gibbons-Hawking term results in a well-posed Dirichlet problem, which is the case relevant to most holographic applications. Following these authors we thus add the term

$$\begin{aligned} S_{\mathrm{YGH}} &= \frac{1}{8\pi G_N} \int_{\partial M} \mathcal{K} \sqrt{|\gamma|} d^d x \\ &= \frac{1}{8\pi G_N} \int_{\partial M} \mathcal{K} d\Sigma \,, \end{aligned} \tag{3.3}$$

i.e. the integral of the (trace of the) second fundamental form, $\mathcal{K}_{ab}$, over the boundary surface ∂M. We use the following notation

$$\gamma_{\mu\nu} : \text{induced metric on } \partial\mathcal{M}$$

$$\mathcal{K} : \text{trace of } 2^{\text{nd}} \text{ fundamental form on } \partial\mathcal{M} \quad (\text{‘extrinsic curvature’}) \,.$$

[1] and hence termed holographic renormalization

We define the second fundamental form via

$$\mathcal{K}_{ab} = \nabla_{(a}n_{b)} \qquad \text{with} \qquad n_b dx^b = k dz\,, \tag{3.4}$$

where k is a normalization constant ensuring $n_a n^a = 1$. Note that $n_a dx^a$ is the ingoing unit normal to a constant z hypersurface. Note also that the hypersurface is timelike, so that the normalization constant k is well defined. Null hypersurfaces are considerably more subtle, but we shall have no need for them in these lectures. In order to learn more about the geometry of Lorentzian embedded hypersurfaces I recommend the book by Eric Poisson as an excellent reference [4].

Worked Exercise: checking the sign of the YGH term

In the literature different conventions for the extrinsic curvature are sometimes employed. In particular some authors define it as the symmetric derivative of the outgoing unit normal, or they may use a different radial coordinate (e.g. $r \sim 1/z$). Accordingly the sign of the Gibbons-Hawking term can be a source of confusion. Furthermore the correct sign choice depends on whether the hypersurface is timelike, null or spacelike[2]. Let us therefore reassure ourselves of the correct sign for the boundary term in our chosen conventions. Furthermore, for simplicity, we do not bother to work in a covariant formalism (we shall do so in full generality below). For now it suffices to check the variational principle for the simplest non-trivial metric. For this purpose the following choice suffices

$$ds^2 = \frac{\ell^2}{z^2}\left(f(z)d\tau^2 + \frac{dz^2}{f(z)} + d\vec{x}^2\right)\,. \tag{3.5}$$

Let us now find the contribution to the on-shell action containing second derivatives of $f(z)$, since this is the troubling term whose variation we

[2]In these notes we are only interested in the case of the AdS boundary, so the hypersurface is always timelike.

want to cancel with the addition of the YGH term. We find

$$S_{\rm EH} = \frac{\beta \mathrm{vol}\left(\mathbb{R}^{d-1}\right)}{16\pi G_N} \int dz \left(\cdots - \left(\frac{\ell}{z}\right)^{d-1} f''(z) \right) , \tag{3.6}$$

the relevant terms being displayed coming from the $\sqrt{-g}R$ part of the action and we have already integrated over all directions but z. Varying with respect to f gives rise to the boundary term

$$\delta S_{\rm EH} = -\frac{\beta \mathrm{vol}\left(\mathbb{R}^{d-1}\right)}{16\pi G_N} \int dz \partial_z \underbrace{\left(\frac{\ell}{z}\right)^{d-1} \delta f'}_{\text{bound. term}} + \cdots \tag{3.7}$$

We see explicitly that the Dirichlet condition on f, namely $\delta f|_{\rm bdry} = 0$ is not sufficient to ensure that this boundary term vanishes. On the other hand, the integral over the second fundamental form gives

$$S_{\rm YGH} := \frac{\beta \mathrm{vol}\left(\mathbb{R}^{d-1}\right)}{8\pi G_N} \mathcal{K} d\Sigma = \frac{\beta \mathrm{vol}\left(\mathbb{R}^{d-1}\right)}{16\pi G_N} \left(\frac{\ell}{z}\right)^{d-1} f' + \cdots \tag{3.8}$$

where the omitted terms do not involve any derivatives acting on $f(z)$. Hence the variation of the YGH term with respect to f gives the following boundary term:

$$\delta S_{\rm GHY} = \frac{\beta \mathrm{vol}\left(\mathbb{R}^{d-1}\right)}{16\pi G_N} \left(\frac{\ell}{z}\right)^{d-1} \delta f' . \tag{3.9}$$

Therefore, we see that the $\delta f'$ boundary terms cancel for the combination

$$S_{\rm EH} + S_{\rm GHY} = \frac{1}{16\pi G_N} \int \sqrt{-g} \left(R + \frac{d(d-1)}{\ell^2} \right) d^{d+1}x + \frac{1}{8\pi G_N} \int \sqrt{-g} \mathcal{K} d^d x , \tag{3.10}$$

which is indeed the sign we chose in our conventions.

We shall now demonstrate in a covariant fashion and in complete generality that the YGH term indeed cancels the relevant piece of the variation of the Einstein-Hilbert term. For this purpose we can neglect the contribution from the cosmological constant as well as the matter fields. The relevant variation of the Einstein-Hilbert action is thus

$$16\pi G_N \delta S_{\mathrm{EH}} = \int \sqrt{-g}\left(R_{ab} - \frac{1}{2}R g_{ab}\right)\delta g^{ab} d^{d+1}x + \int \sqrt{-g} g^{ab}\delta R_{ab} d^{d+1}x\,. \quad (3.11)$$

This variation clearly gives the Einstein equations, provided we can show that the second term is a pure boundary term which furthermore vanishes on ∂M under Dirichlet boundary conditions. Of course we have already anticipated that this will not occur thereby necessitating the addition of the YGH term. But let us now see this explicitly. It is usually easiest to work out variations of geometric quantities, such as δR_{ab} in a local Lorentz frame, that is in a frame defined in the neighborhood of a point where all Christoffel symbols vanish. In such a frame we have

$$\begin{aligned} \delta R_{ab} &= \delta\left(\Gamma^c_{ab,c} - \Gamma^c_{ac,b}\right) \\ &= \left(\delta\Gamma^c_{ab}\right)_{;c} - \left(\delta\Gamma^c_{ac}\right)_{;b}\,. \end{aligned} \quad (3.12)$$

where the second line is a valid rewriting of the first in a local Lorentz frame. As is known from basic (pseudo-) Riemannian geometry, Christoffel symbols themselves are not tensors, but differences of Christoffel symbols are. Thus the variations $\delta\Gamma^a_{cd}$ are tensors and consequently so are their covariant derivatives. Hence our final expression is a tensorial quantity, valid in all frames. Now comes the important part: the variation

$$g^{ab}\delta R_{ab} = V^a{}_{;a} \quad (3.13)$$

is actually a total derivative for the vector

$$V^a = g^{cd}\delta\Gamma^a_{cd} - g^{ab}\delta\Gamma^c_{bc} \quad (3.14)$$

We can thus use the covariant Stokes theorem to write it as an integral over the hypersurface ∂M

$$\int_M V^a{}_{;a}\sqrt{-g}d^{d+1}x = \int_{\partial M} V^a d\Sigma_a = \int_{\partial M} V^a n_a \sqrt{-\gamma} d^d x\,. \quad (3.15)$$

After some algebra (and keeping in mind that $\delta g^{ab} = 0$ everwhere on ∂M, i.e. the

Dirichlet condition) one finds

$$V^a n_a \Big|_{\partial M} = -h^{ab}\delta g_{ab,c} n^c \,. \tag{3.16}$$

where

$$h^{ab} := g^{ab} + n^a n^b \tag{3.17}$$

is the projector onto the constant z hypersurface. After all this work we finally have the covariant expression for the variation of the Einstein-Hilbert action, including the oft-neglected boundary[3] term

$$16\pi G_N \delta S_{\mathrm{EH}} = \int_M \sqrt{-g}\left(R_{ab} - \frac{1}{2}R g_{ab}\right)\delta g^{ab} d^{d+1}x - \int_{\partial M} \gamma^{\mu\nu}\left(\delta g_{\mu\nu,a}\right) n^a \sqrt{-\gamma} d^d x \,. \tag{3.18}$$

It should be kept in mind that this expression is valid only for a timelike hypersurface, the sign of the last term being the opposite in the spacelike case.

We shall now see that the covariant form of the variation of the YGH term is precisely (minus) the final term in this expression. Starting from the definition (3.4), we see that

$$\mathcal{K} = \mathrm{Tr}\mathcal{K}_{ab} = n^a{}_{;a} \tag{3.19}$$

so that its variation is given by

$$\delta\mathcal{K} = \frac{1}{2}h^{cd}\delta g_{cd,a} n^a = \frac{1}{2}\gamma^{\mu\nu}\delta g_{\mu\nu,a} n^a \,. \tag{3.20}$$

It follows immediately that indeed the variation of the York-Gibbons-Hawking term in (3.10) cancels the problematic boundary term in the variation of the Einstein-Hilbert term for the sign chosen there, so long as the hypersurface is timelike, as is the case for the boundary of AdS_{d+1}. For a spacelike hypersurface the sign of the YGH term should be reversed. We are now ready to calculate the on-shell value of the action of AdS_{d+1}. We begin by collecting a few useful relations between curvatures, the cosmological constant and the AdS length ℓ.

[3] A possibly more transparent way of writing the last term in expression (3.18) would have been as $\int h^{cd}\left(\delta g_{cd,a}\right) n^a \sqrt{-h} d^d x$, i.e. in terms of the projector onto the hypersurface $h^{ab} = g^{ab} + n^a n^b$. Of course, restricted to the hypersurface ∂M itself the projector is equal to the induced metric $h^{\mu\nu}\big|_{\partial M} = \gamma^{\mu\nu}$. In the holographic literature it is more customary to talk about $\gamma_{\mu\nu}$ than it is to talk about h_{ab}.

Worked Exercise: Λ for AdS$_{d+1}$

Another source for confusion can be the choice of conventions when denoting the bulk and boundary theories. Here we have chosen to denote the field theory dimension as d and hence the bulk is an asymptotically AdS$_{d+1}$ spacetime. This solves Einstein's equations in $d+1$ with a cosmological constant Λ,

$$R_{ab} - \frac{1}{2} R g_{ab} + \Lambda g_{ab} = 0\,. \tag{3.21}$$

We now fix the relationship between Λ and the AdS length ℓ. We start by taking advantage of the fact that anti-de Sitter space is maximally symmetric, so its Riemann tensor takes the form

$$R_{abcd} = \alpha\left(g_{ac}g_{bd} - g_{ad}g_{bc}\right) \tag{3.22}$$

for some constant α. Taking the trace twice reveals that

$$R = \alpha\, d(d+1)\,. \tag{3.23}$$

Meanwhile the Einstein tensor gives

$$G_{ab} = -\frac{\alpha\, d(d-1)}{2} g_{ab} \tag{3.24}$$

Plugging this into the Einstein equation, we conclude that,

$$\Lambda = \alpha \frac{d(d-1)}{2}\,. \tag{3.25}$$

Finally to relate this to the AdS length we need to calculate an expression for the Ricci scalar in terms of ℓ:

$$ds^2 = \frac{\ell^2}{z^2}\left(dz^2 + \eta_{\mu\nu}dx^\mu dx^\nu\right) \qquad \Rightarrow \quad R = -\frac{d}{\ell^2}(d+1). \tag{3.26}$$

This leads us to the final relation

$$\alpha = -\frac{1}{\ell^2} \qquad \Rightarrow \qquad 2\Lambda = -\frac{d(d-1)}{\ell^2}\,. \tag{3.27}$$

We are now ready to go about calculating the on-shell action for AdS_{d+1}.

3.1.1 Euclidean On-Shell Action for AdS_{d+1}

We want to calculate the actual functional expression of the full action (3.10). For this it is convenient to first use the trace of the Einstein equation (3.21) to find

$$R = -\frac{d(d+1)}{\ell^2}\,. \tag{3.28}$$

So that

$$R + \frac{d(d-1)}{\ell^2} = -\frac{2d}{\ell^2} \qquad \text{while} \qquad \sqrt{-g} = \left(\frac{\ell}{z}\right)^{d+1} \tag{3.29}$$

Putting these facts together one has

$$S_{\mathrm{EH}}\Big|_{o.s.} = -2d\ell^{d-1}\mathrm{vol}(\mathbb{R}^{1,d})\int_{\epsilon}^{\infty}\frac{dz}{z^{d+1}} \propto \frac{1}{\epsilon^d}\,. \tag{3.30}$$

We introduced a cutoff ϵ to regularize the divergence of the integral over the entire AdS_{d+1} spacetime. Note that the divergence comes from the region near the boundary at $z=0$, and as we saw earlier, this region is identified with the UV of the boundary field theory. We should thus think of ϵ as a UV regulator. Up to this point we have been working in Lorentzian signature. In fact, in order to compute the free energy (3.1) we are actually interested in the Euclidean action, obtain by the Wick rotation

$$it = \tau \qquad \text{with} \qquad e^{iS} = e^{-S_E} \tag{3.31}$$

more explicitly, the exponents are

$$iS = i\int \sqrt{-g}(R-2\Lambda)d^{d+1}x = \int \sqrt{g_E}(R_E - 2\Lambda)d^{d+1}x_E\,, \tag{3.32}$$

where the Euclidean expressions are obtained from the metric of Euclidean AdS_{d+1}, that is

$$ds_E^2 = \frac{\ell^2}{z^2}\left(d\tau^2 + dz^2 + \sum_{i=1}^{d-1} dx_i^2\right)\,. \tag{3.33}$$

Euclidean anti-de Sitter space is really just hyperbolic space, $\mathrm{AdS}^E_{d+1} = \mathbb{H}_{d+1}$. Retracing our steps above, we conclude that the Euclidean action has (expectedly) the same degree of divergence as its Lorentzian counterpart. In order to complete our task of computing the on-shell expression of the action, we next need to evaluate the YGH term explicitly.

3.1.2 YGH Term

The UV regulator ϵ we were led to introduce means effectively that space-time is cut-off at a hypersurface at the finite radial position $z = \epsilon$, defining a real boundary of the geometry. We therefore need to evaluate the boundary term on this surface.

The unit normal has components

$$n_a = \frac{\ell}{z}\delta_{az}\,. \tag{3.34}$$

Taking the symmetrized covariant derivative reveals the following non-zero components of the extrinsic curvature:

$$\mathcal{K}_{ab} = \begin{cases} 0 & a \text{ or } b = z \\ -\frac{\ell}{z^2}\delta_{ab} & a, b \neq z \end{cases}. \tag{3.35}$$

Thus the trace takes the value

$$\mathcal{K} = \operatorname{Tr}\mathcal{K}_{ab} = -\frac{d}{\ell}. \tag{3.36}$$

Finally, the metric induced on the cut-off surface is

$$ds^2_{\text{ind}} = \gamma_{\mu\nu}dx^\mu dx^\nu = \frac{\ell^2}{z^2}\left(d\tau^2 + \sum_{i=1}^{d-1} dx_i^2\right)\,. \tag{3.37}$$

Putting all these results together we obtain

$$\int_{\partial M(\epsilon)} \mathcal{K}\sqrt{-\gamma}d^dx = -\frac{d\ell^{d-1}}{z^d}\bigg|_{z=\epsilon} \propto \frac{1}{\epsilon^d} \tag{3.38}$$

The degree of divergence is the same as the Einstein-Hilbert term, but the full calculation revealed that the exact coefficients do not lead to a cancellation of the divergent term between Einstein-Hilbert action and the boundary term. We conclude that the action of AdS^E_{d+1} diverges as

$$S_E(\mathbb{H}_{d+1}) = a_1^{UV}\epsilon^{-d}. \tag{3.39}$$

with a specified coefficient we just determined. We should not be shocked about this, after all we are calculating in quantum field theory, even if it is disguised as a theory of gravity. Therefore, *we need (holographic) renormalization.*

Such a procedure should systematically identify all the divergences and their coefficients and then remove them by adding appropriate counterterms to construct a finite renormalized expression. We shall explore this procedure in detail below, but for now a quick fix can be found by reminding ourselves of the actual physical quantity we want to calculate, namely the free energy of a certain background solution. This could be AdS_{d+1} or the black hole in AdS_{d+1}, for example, or any other equilibrium background we may be interested in. All these have in common that their metric near the boundary approaches that of empty AdS_{d+1}, from which we conclude that their UV divergence structure is universal. Moreover, we know that only free energy differences are physically meaningful. This suggests the use of the background subtraction method, which we illustrate for the case of the black hole inside anti-de Sitter space, i.e. for the case of the thermal free energy of the dual field theory.

Instead of calculating directly its free energy, let us rather ask what its difference in free energy is compared to empty Euclidean anti-de Sitter space, i.e. let us calculate

$$S(BH) - S(\mathbb{H}_{d+1})\,, \tag{3.40}$$

which will turn out to be a finite, well defined quantity. As we saw last time, a Euclidean black hole has a fixed periodicity of its Euclidean time circle, which we will here denote β_2. On the other hand, Euclidean AdS_{d+1} is regular for an arbitrary Euclidean time periodicity, which we will denote as β_2. The question is thus how to fix β_1, which is necessary in order to compare the two metrics to each other.

The correct procedure goes back to work of Hawking and Page [3] and is rather intuitive: one should demand that the two induced metrics agree on the cutoff surface $\partial M(\epsilon)$, i.e. at the radial point $z = \epsilon$. This leads to the equations

$$ds^2_{\text{ind}}(\mathbb{H}_{d+1})\big|_{z=\epsilon} = \frac{\ell^2}{\epsilon^2}\left(d\tau^2 + \sum_i dx_i^2\right) \tag{3.41}$$

$$ds^2_{\text{ind}}(BH)\big|_{z=\epsilon} = \frac{\ell^2}{\epsilon^2}\left(f(\epsilon)d\tau^2 + \sum_i dx_i^2\right). \tag{3.42}$$

Equating these two implies in particular that

$$\beta_1 = \sqrt{f(\epsilon)}\beta_2 = \left(1 - \left(\frac{\epsilon}{z_h}\right)\right)^{1/2}\beta_2 = \left(1 - \frac{1}{2}\left(\frac{\epsilon}{z_h}\right)^d + \cdots\right)\beta_2. \tag{3.43}$$

With the help of this relation we find

$$S(BH) - S(\mathbb{H}_{d+1}) = -\frac{\beta_2 \text{vol}\left(\mathbb{R}^{d-1}\right)}{16\pi G_N}\int_{z_h}^{\epsilon}\frac{2d\ell^{d-1}}{z^{d+1}}dz + \frac{\beta_1 \text{vol}\left(\mathbb{R}^{d-1}\right)}{16\pi G_N}\int_{\infty}^{\epsilon}\frac{2d\ell^{d-1}}{z^{d+1}}dz\,. \tag{3.44}$$

Notice that the two terms, while involving the exact same integrand, differ in a few respects. Firstly we have the two different β parameters, and secondly we have different limits of integration, in one case over all of AdS_{d+1} and in the other from the horizon radius to the boundary. We can use the same trick as in section 3.1.1 to calculate the Einstein-Hilbert part of the on-shell action for the black hole. We have

$$S_{\text{EH}}(BH) = -\frac{1}{16\pi G_N}\int\sqrt{g_E}\underbrace{\left(R_E + \frac{d(d-1)}{\ell^2}\right)}_{\substack{=\text{ same on-shell}\\ \text{expression as } AdS_{d+1}}}d^{d+1}x\,. \tag{3.45}$$

This implies that all UV divergence cancel as needed. However, in addition we get two finite contributions. The first is the horizon contribution in the first integral of Eq. (3.44), and the second comes from using the expansion of the inverse temperature (3.43) in the second integral. In detail,

$$\frac{16\pi G_N}{2d\ell^{d-1}\beta_2 \mathrm{vol}\left(\mathbb{R}^{d-1}\right)}\left[S(BH)-S(\mathbb{H}_{d+1})\right]=-\underbrace{\int_{z_h}^{\epsilon}}_{①}\frac{dz}{z^{d+1}}-\int_{\infty}^{\epsilon}\frac{dz}{z^{d+1}}\Big(1-\underbrace{\frac{1}{2}\Big(\frac{\epsilon}{z_h}\Big)^{d}}_{②}\Big)\,, \tag{3.46}$$

We denoted by ① the contribution from the horizon, which is essentially the entropy, and by ② the contribution from the finite term at the boundary, which is essentially the (ADM) energy. Thus,

$$\begin{aligned}\frac{F}{\mathrm{vol}\left(\mathbb{R}^{d-1}\right)}=\frac{S^E_{BH}-S_{\mathbb{H}_{d+1}}}{\beta_2 \mathrm{vol}\left(\mathbb{R}^{d-1}\right)}&=-\frac{\ell^{d-1}}{8\pi G_N}\frac{d}{z_h^d}+\frac{\ell^{d-1}}{16\pi G_N}\frac{d}{z_h^d}\\&=\varepsilon-sT\end{aligned} \tag{3.47}$$

In the second line we have written the two contributions to the free energy in their standard form, using

$$S=\frac{A}{4G_N}=\frac{\mathrm{vol}\left(\mathbb{R}^{d-1}\right)}{4G_N}\left(\frac{\ell}{z_h}\right)^{d-1} \qquad \text{so that} \qquad s:=\frac{S}{\mathrm{vol}\left(\mathbb{R}^{d-1}\right)} \tag{3.48}$$

and

$$\beta_2^{-1}:=T=\frac{d}{4\pi z_h}\,, \tag{3.49}$$

which follows from demanding regularity of the Euclidean manifold at z_h as discussed last time. The last term, denoted by ε can be checked to be equal to the ADM energy density. Here we have established this relation for but one example, namely that of the black hole in anti-de Sitter space. Using the Euclidean techniques outlined in this lecture one can actually prove a free energy relation of this form quite generally. In the present case the free energy can be simplified to give

$$\frac{F}{\mathrm{vol}\left(\mathbb{R}^{d-1}\right)}=-\frac{(4\pi)^d\ell^{d-1}}{16\pi G_N d^d}T^d, \tag{3.50}$$

which coincides with the relation claimed in the previous lecture. Let us conclude by listing some comments.

Comments

1. As alluded to in the previous paragraph, one can develop this into a fully fledged derivation of BH thermodynamics in AdS[4]. This structure is a robust output of the saddle-point approximation to Euclidean quantum gravity. We should think of this as a feature of gravity as an effective field theory and therefore accept its validity in the semi-classical regime of any theory of quantum gravity.

2. Our calculation of the planar black hole has revealed that the free energy (3.50) has a definite sign. This is not the case in global AdS. For example, in global AdS_4, we have $F = c\frac{\ell^2 - r_+^2}{\ell^2 + 3r_+^2}$ for some constant $c > 0$ whose exact value is of no import here. Evidently this can change sign, according to

$$\begin{aligned} F > 0 \qquad &\text{for} \qquad r_+ < \ell \\ F < 0 \qquad &\text{for} \qquad r_+ > \ell \end{aligned} \tag{3.51}$$

   Since we are really looking at the difference in free energy between the black hole and thermal AdS, this means that depending on the value of r_+/ℓ one or the other will be the preferred solution. Translated into a critical temperature, T_{HP}, one finds that the black hole is preferred for temperatures higher than T_{HP} and thermal AdS for temperatures below T_{HP}. This is called the 'Hawking-Page' transition and it was reinterpreted in the context of holography as the confinement / de-confinement transition in the dual gauge theory.

3. Our regularization was rather *ad-hoc*, dealing only with one kind of divergence and focussing only on free energy. One might ask for example what happens with $n-$point. functions. We clearly need to systematize this procedure, which will lead us to a more in-depth discussion of *holographic renormalization*

The holographic renormalization procedure basically follows the same steps as the ordinary field-theory case. We conclude by briefly outlining the steps we will discuss in more details below:

[4]One can also do this for asymptotically flat black holes, but this is not the subject of the present discussion

(I) Define what we mean by an 'asymptotically AdS' spacetime. In the field theory this amounts to carefully defining the UV fixed point.

(II) Regulate divergences of AdS integrals resulting from calculations of n-pt functions and free energies. These should be universal and only depend on the UV structure of the solutions.

(III) Introduce suitable counterterms at UV boundary to cancel the universal divergences.

(IV) Define finite, renormalized $n-$point functions via a renormalized generating functional. This functional also gives the free energy in the thermal case.

Bibliography

[1] V. Balasubramanian and P. Kraus, *Spacetime and the Holographic Renormalization Group*, *Physical Review D* **15** (1999) pg. 3605-3608. arxiv:9903190.

[2] K. Skenderis, *Lecture notes on holographic renormalization*, arxiv:0209067.

[3] S. W. Hawking and D. N. Page, *"Thermodynamics of Black Holes in Anti-de Sitter Space,"* Commun. Math. Phys. **87** (1983) 577. doi:10.1007/BF01208266

[4] E. Poisson, *"A relativisit's toolkit: the mathematics of black-hole mechanics"*, Cambridge university press (2004)

Chapter 4

Correlations

4.1 Correlation Functions: General Issues

Up to now we have mostly dealt with properties of equilibrium partition functions, such as thermodynamic free energies and related quantities. In many situations, be they concerned with applications or matters of principle, we will also want to understand time dependent notions. In quantum field theory this usually involves determining correlation functions in Lorentzian signature, taken in states which are not eigenstates of the Hamiltonian. The situation is no different in holography. Let us therefore spend some time to establish the basics of (Lorentzian) correlation functions. We will start with very general issues before spending some time on correlations in equilibrium. At the end we will consider the most general situation, namely correlations in nonequilibrium states, or ensembles of nonequilibrium states.

In Euclidean signature $n-$point functions,

$$G^E(x_1, \ldots, x_n) = \langle \mathcal{O}(x_1)\mathcal{O}(x_2)\cdots\mathcal{O}(x_n)\rangle \,, \tag{4.1}$$

do not depend[1] on operator ordering. This follows from the fact that operators at

[1] Nevertheless the Euclidean path integral produces correlation function in Euclidean time order

$$\int \mathcal{D}\phi \; \mathcal{O}(x_1)\cdots\mathcal{O}(x_n)e^{-S_E} \sim \langle T_E\mathcal{O}(x_1)\cdots\mathcal{O}(x_n)\rangle \,.$$

separated points commute

$$[\mathcal{O}(x), \mathcal{O}(y)] = 0 \qquad\qquad \text{for } x \neq y \text{ in Euclidean}$$

While in Lorentzian signature this is only true at space-like separation

$$[\mathcal{O}(x), \mathcal{O}(y)] \qquad \begin{cases} = \ 0 & x - y \qquad \text{is spacelike} \\ \neq \ 0 & x - y \qquad \text{timelike or null} \end{cases}$$

In Lorentzian signature, therefore, the ordering matters. For this reason we can define a number of different correlation functions. For example, at the level of two-point functions, we have the well-known choices of time-ordered, advanced and retarded correlation functions,

$$\begin{aligned} iG^T(x_1, x_2) &= \langle T\, \mathcal{O}(x_1)\mathcal{O}(x_2)\rangle \\ iG^A(x_1, x_2) &= \Theta(x_2^0 - x_1^0)\langle[\mathcal{O}(x_2), \mathcal{O}(x_1)]\rangle \\ iG^R(x_1, x_2) &= \Theta(x_1^0 - x_2^0)\langle[\mathcal{O}(x_1), \mathcal{O}(x_2)]\rangle. \end{aligned} \tag{4.2}$$

Note that these correlation functions depend on the state they are evaluated in, which could be for example the vacuum or a thermal ensemble. In the zero-temperature case, that is the vacuum, the Lorentzian correlation functions can be obtained from the former by analytic continuation, e.g. via the $\epsilon-$convention (Osterwalder-Schrader reconstruction). This procedure can get quite complicated for higher-point functions, as we shall see. At finite temperature the continuation from Matsubara Green functions is even more complicated. Lastly, if correlators are being calculated numerically, as is often the case in holography, analytic continuation is very tricky indeed. For these reasons it is important to have a real-time prescription for calculating correlation functions in AdS/CFT. Luckily such a formalism is available and we will now turn to describing it. We shall briefly remind ourselves of the Euclidean case and then look at equilibrium in Lorentzian signature. In a later chapter we treat the general case away from equilibrium and in

In Euclidean signature we have a choice as to what we call Euclidean 'time', and the ordering depends on this choice. In radial quantization, for example, this produces radially ordered correlation functions.

Lorenztian signature.

4.2 Correlation Functions: How To Calculate Them

4.2.1 Recap of GKPW Prescription

Let us now review the procedure to calculate correlation functions in Euclidean signature, due to Gubser, Klebanov, Polyakov and Witten (GKPW), [2, 3]. These authors proposed a relationship between the gravity partition function in AdS_{d+1} and the generating functional for connected field theory correlation functions.

$$Z_{\text{gravity}}[g_{(0)}, \phi_{(0)}, \ldots] \equiv \langle e^{-\int \phi_{(0)} \mathcal{O}_\phi d^d x + \cdots} \rangle_{\text{QFT}} \ . \tag{4.3}$$

This relation says that the gravity partition function where the fields asymptote to specific functions at the boundary (e.g. $\Phi \to \phi_{(0)}$) is equal to the QFT path integral where we insert a source $\phi_{(0)}$ for an operator $\mathcal{O}_\phi$. We define the gravity partition function somewhat symbolically via the path integral over all metrics and field configurations

$$Z_{\text{gravity}}[g_{(0)}, \phi_{(0)}, \ldots] = \int \mathcal{D}g \mathcal{D}\phi e^{-S_E[g,\phi]} \tag{4.4}$$

with specified boundary behavior $\phi_{(0)}$ and $g_{(0)}$. We have already seen that we can make sense of this object to one loop in a semi-classical expansion around a background solution [1]. For the purpose of this course this is all we need, but ultimately this object needs to be embedded in a UV complete theory of gravity if it is to have an independently well-defined meaning.

Let us pause briefly to make a philosophical comment. One of the most powerful aspects of holography is that we can actually view the dual field theory as giving a non-perturbative UV complete definition of the gravity partition function. That is we take the right hand side of equation (4.3) as the definition of what we mean by quantum gravity on asymptotically anti-de Sitter spaces. It is in this sense that AdS/CFT is celebrated as such a significant development in quantum gravity.

Let us return to the technical meaning of Eq. (4.3). We have already seen that in the semiclassical limit

$$Z_{\text{gravity}} = \det\left[\delta^2 S\right] e^{-S^E_{\text{on-shell}}} \,. \tag{4.5}$$

The prefactor denotes the functional determinant of the quadratic kernel of the action around the semi-classical solution. Putting this relation together with the equivalence (4.3) we conclude that the generating functional of connected QFT correlations is given by the Euclidean on-shell action

$$S^E_{\text{on-shell}}[\phi_{(0)}] = -W_{\text{QFT}}[\phi_{(0)}] \,. \tag{4.6}$$

From this it follows immediately that Euclidean n-point functions are obtained by functional differentiating of the gravity partition function with respect to its boundary values

$$\begin{aligned} \langle \mathcal{O}(x) \rangle &= \left. \frac{\delta S_{\text{on-shell}}}{\delta \phi_{(0)}(x)} \right|_{\phi_{(0)}=0} \\ \langle \mathcal{O}(x_1) \cdots \mathcal{O}(x_n) \rangle &= (-1)^{n+1} \left. \frac{\delta^n S_{\text{on-shell}}}{\delta \phi_{(0)}(x_1) \cdots \delta \phi_{(0)}(x_n)} \right|_{\phi_{(0)}=0} \,. \end{aligned} \tag{4.7}$$

This procedure seems to have a flaw, however. Note that the bulk fields typically obey second order equations of motion, but we only specify one boundary condition, namely $\Phi \to \phi_{(0)}$. This means that so far the bulk solution and thus the partition function aren't uniquely specified. However, in Euclidean signature we also need to make sure that the solutions are regular which leads to a second, so-called 'interior', boundary condition on the fields. It turns out that one of the independent solutions blows up at the IR end of the bulk geometry and so we are forced to discard it. Thus the remaining freedom is fixed by *regularity* in Euclidean signature.

Worked Example: Euclidean 2-point function

Euclidean two point function for scalar operator (details...)

This, however, is no longer true in Lorentzian signature, where what used to be the divergent and and decaying solutions in the interior, now both are oscillatory, not distinguished from each other, at sight, by any regularity condition. Replacing the diverging and decaying behavior we now have ingoing solutions and outgoing solutions. Since neither is more regular than the other we are also free to choose linear combinations of the two. Intuition then suggests that this choice of interior boundary conditions should correspond to the different analytic continuations of the Euclidean correlation function

$$G^{E}\left(x_1, \ldots x_n\right) = \left\langle \mathcal{O}(x_1) \cdots \mathcal{O}(x_n) \right\rangle,$$

which as we saw above leads to the different time orderings in Lorentzian correlation functions. Let us now see this explicitly for the example of the $2-$point function. We will firstly state the procedure to obtain this quantity and then outline a more general framework that gives rise to this prescription and that also allows generalization to higher-point functions. We remind the reader that for now we are restricting ourselves to correlations at or near equilibrium, postponing the more general case until later.

4.2.2 Lorentzian Two-point Function: Recipe

For simplicity, let us work with a scalar operator $\mathcal{O}_\phi(x^\mu)$ dual to the bulk field $\phi(z, x^\mu)$ which will obey the simplest possible equation of motion, that of a free massive scalar. We have

$$\mathcal{O}_\phi \leftrightsquigarrow \phi(z, x^\mu) \tag{4.8}$$

Let us write our background in the form

$$ds^2 = g_{zz}(z)dz^2 + g_{\mu\nu}(z)dx^\mu dx^\nu \tag{4.9}$$

and let us recall that we use mostly plus signature for the metric. In this case we can write the action of the scalar field as

$$S[\phi] = -K \int d^d x \int_{z_h}^{z_{UV}} dz \sqrt{-g} \left[g^{zz} (\partial_z \phi)^2 + g^{\mu\nu} \partial_\mu \partial_\nu \phi + m^2 \phi^2 \right] , \tag{4.10}$$

where K is some normalization factor. This action leads to the equation of motion

$$\frac{1}{\sqrt{-g}} \partial_z \left(\sqrt{-g} g^{zz} \partial_z \phi \right) + g^{\mu\nu} \partial_\mu \partial_\nu \phi - m^2 \phi = 0 \, . \tag{4.11}$$

Depending on the details of the background metric[2] an analytic solution for the field ϕ may or may not be available. In fact, in most cases this is not the case, and so we shall outline the procedure in a general fashion without the need for an explicit solution of the equations of motion. Any such solution will have a certain asymptotic behavior near the boundary of AdS, i.e. in the UV region of the field theory. We shall return to this below. However, as noted above, we must also specify a boundary condition at the bulk, or 'infra-red' end of the geometry, which we denote by z_h. This suggests an interpretation as a horizon, and indeed often this is the case, but other cases exists, such as for example the hard-wall in holographic duals to gapped theories [4]. We now present a four-step procedure that allows the determination of the two point function [5].

1. We start by writing the solution in terms of Fourier modes[3] as follows

$$\phi(z, x) = \int \frac{d^d k}{(2\pi)^d} e^{ik \cdot x} f_k(z) \phi_0(k), \tag{4.12}$$

where $f_k(z)$ is the bulk-boundary propagator, a solution to the equations of motion (4.11) with a delta function source localized to a single point at the boundary (in real space). In Fourier space this is a solution of the equation

$$\frac{1}{\sqrt{-g}} \partial_z \left(\sqrt{-g} g^{zz} \partial_z f_k \right) - \left(k^\mu k_\mu + m^2 \right) f_k = 0 \, , \tag{4.13}$$

[2]We have allowed for a certain generality in formulating our metric ansatz, but obviously we are not capturing the most general case. More on this in later chapter.

[3]We remark that this is possible because of our rather symmetric choice of background solution. Typically the ground states will satisfy such a symmetry, but if one considers time-dependent backgrounds everything becomes a lot messier, as you can easily imagine.

which asymptotes to unity

$$f_k(z) \overset{z\to z_{UV}}{\longrightarrow} 1 \tag{4.14}$$

as z approaches the boundary. One can check that now the field $\phi(z, x^\mu)$ indeed asymptotes to the boundary behavior $\phi_0(x^\mu) = \int e^{ik_\mu x^\mu}\phi_0(k^\mu)d^dk$ at $z_{_{UV}}$. We now have completely solved the ultraviolet part of our problem.

2. The function $f_k(z)$ still contains two different behaviors at the infra-red end, z_h. We now fix this freedom by demanding a certain behavior at z_h. This is to say that we make a choice whether the function f_k is purely ingoing at z_h or purely outgoing or a combination of the two. It is important to emphasize that at this point we must take into account some physical input as to which choice is appropriate. As we will see, this is exactly the same as the usual situation, where it is also up to us whether we want to work with the advanced Green function, the retarded Green function, or a combination, such as the Feynman Green function. This choice fixes the infra-red part of our solution. We now have a fully specified solution of the equations of motion of the form (4.12) which obeys the correct asymptotic behavior at both ends.

3. Our next step is to calculate the onshell value of the action (4.10). A calculation shows that this takes the form

$$S_{\text{on-shell}} = \int \frac{d^dk}{(2\pi)^d}\phi_0(-k)\mathcal{F}(k,z)\phi_0(k) \tag{4.15}$$

where we define $\mathcal{F}(k,z) = K\sqrt{-g}g^{zz}f_k^*(z)\partial_z f_k(z)$, the so-called 'flux factor'. The point is that we have now succeeded in expressing the onshell action in terms of the sources, which is the form appropriate for a generating functional for correlations. And in the Euclidean case this would be exactly what we have, so that correlation functions would be obtained by successive derivatives with respect to the source. However, in Lorentzian signature this is not the case. Let us remark that below we will outline a method which gives rise to such a function also in the Lorentzian case, but this involves switching to the so-called Schwinger Keldysh formulation of QFT, a complication that is in practice often unnecessary and that we side step for now.

4. We now state that the Lorentzian Green function is given by the expression

$$G^{R/A}(k) = -2 \lim_{z \to z_{\mathrm{UV}}} \mathcal{F}(k, z) \tag{4.16}$$

where we have indicated once more that the exact nature of the Green function depends on our choice of infra-red boundary condition on f_k and that in principle this allows for advanced behavior as well as retarded behavior and a combination thereof. This completes the four-step procedure to compute Lorentzian 2-point functions.

The approach above is computationally very convenient and it gives the right expressions for the Green functions, but from what we said, it is not actually clear why the procedure works and also how one would extend it to higher n point function. Let us now address these issues.

General Procedure

We will now outline the more general approach which a) explains the origin of the prescription above in a satisfactory manner and b) allows a generalization to arbitrary n-point functions. We will however restrict ourselves to the case where Green functions near equilibrium are computed. An even more general approach exists and we will return to this issue later on.

A first hint that something more general is needed comes from noting that the above Green function, Eq. (4.16) does not follow from treating (4.15) as a generating functional for correlations. Let us assume, for argument's sake, that we made such an assignment,

$$S^{\mathrm{Lor.}}_{\text{on-shell}}[\phi_{(0)}] = -W^{\mathrm{Lor.}}_{\mathrm{QFT}}[\phi_{(0)}]\,, \tag{4.17}$$

Then taking two derivatives with respect to the source gives

$$\langle \mathcal{O}(k)\mathcal{O}(-k)\rangle := \frac{1}{i^2}\frac{\delta^2 S^{\mathrm{Lor.}}_{\text{on-shell}}}{\delta\phi_{(0)}(k)\delta\phi_{(0)}(-k)} = -\left.\mathcal{F}(k,z)\right|^{z_h}_{z_{UV}} - \left.\mathcal{F}(-k,z)\right|^{z_h}_{z_{UV}}\,, \tag{4.18}$$

which picks up a contribution from the horizon, and which is not the same as the correct answer (4.16). So we need to replace the naive generating functional (4.17) with something different. Such a quantity can be constructed in terms of a Schwinger-Keldysh type prescription for AdS. One way to arrive at this, in the present context, is to take gravity by its word and take seriously the fact that an equilibrium black hole solution has two asymptotic regions[4]. What is meant here, of course, is the maximal analytic extension of the black hole geometry due to Kruskal and others, as shown in Fig. 4.1. For such an anti-de Sitter black hole, each asymptotic region has its down asymptotic boundary, and so it is only natural to consider the dual field theory as defined on both. This means, however, that when talking about bulk solutions we now must specify their asymptotic behavior on each boundary. That is we have

$$\begin{aligned} \phi(z, x^\mu) &\longrightarrow \phi^{(1)}_{(0)}(x^\mu) \qquad z \to \text{ boundary 1} \\ \phi(z, x^\mu) &\longrightarrow \phi^{(2)}_{(0)}(x^\mu) \qquad z \to \text{ boundary 2} \end{aligned} \tag{4.19}$$

Now we have an onshell action that depends on the sources on both boundaries,

$$S^{\text{Lor.}}_{\text{on-shell}}\left[\phi^{(1)}_{(0)}, \phi^{(2)}_{(0)}\right] = S\left[\phi^{(1)}_{(0)}\right] - S\left[\phi^{(2)}_{(0)}\right] . \tag{4.20}$$

The sign in front of $S\left[\phi^{(2)}_{(0)}\right]$ follows because the natural time-like Killing vector in the second asymptotic region in fact generates backward time evolution when compared to the first asymptotic region. This also is a strong clue. Apparently we have an object which incorporates normal time evolution on boundary 1, where we are free to insert sources $\phi^{(1)}_{(0)}$ and also backwards time evolution on boundary 2, where we are free to insert sources $\phi^{(2)}_{(0)}$. This is exactly the structure arising in the two-time or Schwinger-Keldysh formulation of quantum field theory (reviewed in Chapter 5). Now we are of course also free to consider functional derivatives of the new generating functional with respect to sources on both boundaries, that is we have a matrix

$$iG_{ab}(x,y) = \frac{1}{i^2}\frac{\delta^2 \log Z[\phi^{(1)}_{(0)}, \phi^{(2)}_{(0)}]}{\delta\phi_a(x)\delta\phi_a(y)} = i\begin{pmatrix} G_{11} & -G_{12} \\ -G_{21} & G_{22} \end{pmatrix}_{ab} \tag{4.21}$$

[4] Again, we emphasize this is only true near equilibrium. We shall see a little later what should replace the other side of the geometry when we are far from equilibrium.

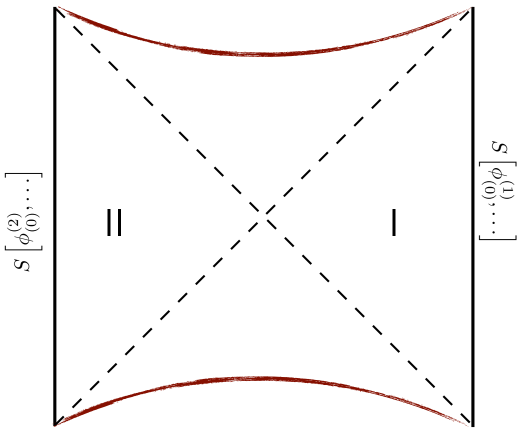


Figure 4.1: Maximal extension of the AdS black hole geometry. We see that this manifold now has two asymptotically AdS boundaries, one in region I and the other in region II. In order to fully define the bulk problem we need to specify the boundary conditions on each one, as well as insist on a good analytical extension of modes through the bifurcation linking the horizons. This naturally leads to a matrix of correlation functions as in Eq. (4.22) at temperature $\beta = \frac{1}{T_H}$. This is the holographic equivalent of the Schwinger-Keldysh formulation of nonequilibrium quantum field theory.

In more detail we have [4]

$$\begin{aligned} G_{11}(k) &= \mathrm{Re}\, G_R(k) + i \coth\frac{\omega}{2T}\, \mathrm{Im}\, G_R(k) \\ G_{12}(k) &= \frac{2ie^{-(\beta-\sigma)\omega}}{1-e^{-\beta\omega}} \mathrm{Im}\, G_R(k) \\ G_{21}(k) &= \frac{2ie^{-\sigma\omega}}{1-e^{-\beta\omega}} \mathrm{Im}\, G_R(k) \\ G_{22}(k) &= -\mathrm{Re}\, G_R(k) + i \coth\frac{\omega}{2T}\, \mathrm{Im} G_R(k)\,. \end{aligned} \tag{4.22}$$

Here we have introduced an arbitrary parameter $\sigma \in (0, \beta]$ which is convention dependent.[5] A particularly nice convention is the symmetric choice $G_{12} = G_{21}$ that is $\sigma = \beta/2$ in Eq. (4.22). One sometimes sees other choices, and indeed we will later encounter such an occasion. The upshot is that, whatever the convention, in each case the different matrix elements of (4.21) contain the same physical

[5] Let us now go into the details of this somewhat annoying convention dependence here. We will return to this and more details of the two-time approach later.

information. The main point is that we can extract G_R and $G_A = G_R^\dagger$ from this in a way that reproduces the result obtained above from the somewhat ad-hoc prescription, but now in an fully consistent formalism which includes a generating functional of connected Lorentzian correlation functions, as in Eq. (4.21). Notice that now we can, in principle, take higher-order derivatives

$$G^{(n)}_{a_1 a_2 \cdots a_n}(x_1, x_2, \ldots x_n) = \frac{1}{i^n} \frac{\delta^2 \log Z[\phi^{(1)}_{(0)}, \phi^{(2)}_{(0)}]}{\delta\phi_{a_1}(x_1)\delta\phi_{a_2}(x_2)\cdots\delta\phi_{a_n}(x_n))} \tag{4.23}$$

to obtain higher n-point functions. We see that now the index structure becomes increasingly complicated, but this is as it has to be, since more and more choices of time ordering appear, the higher we go up in the order of a correlation function. Again, in principle, all these Lorentzian correlation functions are calculable as different analytic continuations of the Euclidean n-point functions, but again here we face the same plethora of choices concerning the ordering of Lorentzian operators. It is certainly good to have a systematic, intrinsically Lorentzian prescription, as outlined here.

4.3 Example: BTZ Black Hole or Thermal Correlators in CFT_2

We illustrate the abstract discussion above by going in detail through a concrete example. We choose the background to be the three-dimensional black hole or 'BTZ' black hole. This will give rise to thermal correlation functions of a two-dimensional CFT. It is one of the few non-trivial examples where a fully analytic treatment is possible. Nevertheless the answers exhibit the full intricate structure of more complicated examples. We will therefore learn quite a bit about the general features of correlation functions in holography from this exercise.

Let us begin by laying down some definitions. We work with the BTZ metric in the form

$$ds^2 = \ell^2 d\mu^2 - \sinh^2\mu(dx^+)^2 + \cosh^2(dx_-)^2\,. \tag{4.24}$$

This metric is a solution of the Einstein equations with negative cosmological constant, where in our conventions $d = 2$ and $\Lambda = -1/\ell^2$. This rather convenient form of the metric can be rewritten in terms of the more conventional coordinates

$$\rho^2 = \rho_+^2 \cosh^2 \mu - \rho_-^2 \sinh^2 \mu \,, \qquad x^\pm = \pm \rho_\pm t \mp \rho_\mp \varphi \,. \tag{4.25}$$

The metric becomes

$$ds^2 = -\frac{\left(\rho^2 - \rho_+^2\right)\left(\rho^2 - \rho_-^2\right)}{\rho^2} dt^2 + \rho^2 \left(d\varphi - \frac{\rho_+ \rho_-}{\rho^2} dt \right)^2 + \frac{\rho^2}{\left(\rho^2 - \rho_+^2\right)\left(\rho^2 - \rho_-^2\right)} d\rho^2. \tag{4.26}$$

This is the familiar form of the metric of a non-extremal BTZ black hole with an outer horizon at ρ_+ and an inner horizon at ρ_-. We now consider an operator in the dual CFT with conformal dimension Δ and spin s, dual to a bulk field ϕ. In two-dimensional conformal field theory it is customary to rewrite the dimensions in terms of so-called left and right-moving weights

$$(h_L + h_R) = \Delta \,, \qquad (h_L - h_R) = s \,. \tag{4.27}$$

To simplify matters, let us work with the spinless case $s = 0$, that is $h_L = h_R = h$. This means that the dual bulk field, ϕ, is a scalar of mass

$$\Delta = 1 + \sqrt{1 + m^2 \ell^2} \,. \tag{4.28}$$

Worked Exercise: The Wave Equation in BTZ

Let us perform one further change of coordinates, namely let us change the radial coordinate to $z = \tanh^2 \mu$. We have

$$dz = 2\,\mathrm{sech}^2 \mu \tanh \mu d\mu = 2(1 - z)\sqrt{z} d\mu \tag{4.29}$$

On the other hand,

$$\begin{aligned}\sinh^2\mu &= z\cosh^2\mu = \frac{z}{1-z} \\ \cosh^2\mu &= \frac{1}{z}\sinh^2\mu = \frac{1}{1-z}\end{aligned} \tag{4.30}$$

and thus the metric in these coordinates becomes

$$ds^2 = \frac{\ell^2}{4z(1-z)^2}dz^2 - \frac{z}{1-z}dx_+^2 + \frac{1}{1-z}dx_-^2\,. \tag{4.31}$$

We can now write the wave equation for a scalar field $\phi(z, x^\mu)$ of mass m in this background,

$$\frac{1}{\sqrt{-g}}\partial_a\left(\sqrt{-g}g^{ab}\partial_b\phi\right) = m^2\phi\,. \tag{4.32}$$

Writing the terms out explicitly, we have

$$\partial_z^2\phi + \frac{1}{z}\partial_z\phi + \frac{\ell^2}{4z(1-z)^2}\left[\Box_{(\mu)} - m^2\right]\phi = 0\,. \tag{4.33}$$

Here $\Box_{(\mu)}$ denotes the d'Alambertian in the boundary directions x^μ. Evidently upon Fourier transforming, $\Box_{(\mu)} \to -k^2$ and the momentum will enter the equation like a mass term.

We are now ready to follow the recipe of section 4.2.2.

We want to solve the equations of motion of a massive scalar field in the background (4.24). For this purpose, let us choose a rescaled radial coordinate $z = \tanh^2\mu$ and let us parametrize the solution as follows

$$\phi(x^\pm, z) = \int \frac{dk_+dk_-}{(2\pi)^2}e^{-i(k_+x^+ + k_-x^-)}f_k(z, k_+, k_-)\phi_{(0)}(k_+, k_-)\,, \tag{4.34}$$

where we have defined the mode function $f_k(z, k_+, k_-)$, which we also often write more succinctly as $f_k(z)$, reminding the reader of its momentum dependence only through its subscript. This solves the massive wave equation in the BTZ black whole, so long as f_k satisfies the hypergeometric equation

$$\partial_z^2 f_k + \frac{1}{z}\partial_z f_k + \left[\frac{\ell^2 k_+^2}{4z^2(1-z)} - \frac{\ell^2 k_-^2}{4z(1-z)} - \frac{m^2\ell^2}{4z(1-z)^2}\right] f_k = 0. \tag{4.35}$$

According to the recipe, we need the flux factor

$$\mathcal{F}(k_+, k_-, z) = K\frac{2z}{\ell} f_k^* \partial_z f_k, \tag{4.36}$$

from which we obtain the Lorentzian two-point function as

$$G(k_+, k_-) = -2 \lim_{z\to z_{UV}} \mathcal{F}(k_+, k_-, z). \tag{4.37}$$

We should choose f_k purely *ingoing* for the *retarded* Green function and purely *outgoing* for the *advanced* Green function. The solution to (4.35) is given by

$$f_k(k_+, k_-) = \mathcal{N} z^{\alpha}(1-z)^{\beta}\, {}_2F_1(a, b, c, z)\,, \tag{4.38}$$

where

$$\begin{aligned} \alpha &= \pm\frac{i\ell k_+}{2}\,, \qquad \beta = \frac{1}{2}\left(1 - \sqrt{1 + m^2\ell^2}\right), \\ a &= \frac{\ell k_+ - \ell k_-}{2i} + \beta, \qquad b = \frac{\ell k_+ + \ell k_-}{2i} + \beta, \qquad c = 1 + 2\alpha. \end{aligned} \tag{4.39}$$

This sign choice for α corresponds to the ingoing or outgoing solution, as we shall see in detail below. We also remark that the sign on β differs from the one in the definition of Δ above. The constant $\mathcal{N}$ is a normalization factor such that $f_k \to 1$ as $z \to z_{UV}$. Explicitly

$$\mathcal{N}^{-1} = z_{\rm UV}^{\alpha}(1 - z_{\rm UV})^{\beta}\, {}_2F_1(a, b, c, z_{\rm UV})\,. \tag{4.40}$$

4.3.1 Asymptotics of the Wave Equation

As we just established the solution of the wave equation in BTZ can be written in terms of hypergeometric functions

$$f_k = \frac{z^\alpha (1-z)^\beta \, {}_2F_1(a,b,c,z)}{z_{\mathrm{UV}}^\alpha (1-z_{\mathrm{UV}})^\beta \, {}_2F_1(a,b,c,z_{\mathrm{UV}})} \,. \tag{4.41}$$

This expression incorporates near-horizon and near-boundary asymptotics of the wave equation. We shall now investigate these in a little more detail.

Near-horizon Behavior

We now expand this near the horizon, using the standard expressions[6] for ${}_2F_1[a,b,c;z]$. We find

$$z^\alpha (1-z)^\beta {}_2F_1[a,b,c;z] \approx z^\alpha \left(1 + \left(\frac{ab}{c} - \beta\right) z + \cdots \right). \tag{4.42}$$

Thus, to leading order in z, the mode function behaves as

$$\begin{aligned} f_k(z) &\sim \exp\left(-ik_+ x^+ - ik_- x^-\right) z^\alpha \\ &= \exp\left(-ik_+ x^+ - ik_- x^- \pm \frac{ik_+}{2} \log z\right) \\ &= \exp\left(-ik_+ \left(x^+ \mp \frac{1}{2} \log z\right) - ik_- x^-\right). \end{aligned}$$

We now consider the motion of a wavefront in the radial direction. We know that x^+ increases as time increases and so, for the upper sign, $\log z$ must also increase in order to stay on surfaces of constant phase, i.e. on the wavefronts. This means that z itself increases. Therefore the wavefront moves in the direction of increasing z which means it moves from the horizon at $z = 0$ towards the boundary at $z = 1$. This is what we call an outgoing mode. By exactly the same reasoning, for the lower sign the wave front is ingoing into the horizon. We thus conclude that

$$\alpha = -\frac{ik_+}{2} \ell \tag{4.43}$$

[6] These can be looked up in integral tables, such as Abramowitz and Stegun, or by consulting Mathematica or a similar software.

is the exponent corresponding to the ingoing solution.

Near-boundary Behavior

The goal is to find the flux factor $\mathcal{F}$, (4.16), and then evaluate it in the boundary limit $z \to z_{\mathrm{UV}}$. For this it evidently suffices to determine the leading behavior as z approaches the boundary. We shall now do so. The point $z = 1$ is a regular singular point of the differential equation (4.35). We may thus look for a Frobenius type series of the form $(1-z)^{\beta}$. The indicial equation tells us then that the solution is

$$\phi(z) \sim A(k)(1-z)^{\beta_-}\left(1+\cdots\right) + B(k)(1-z)^{\beta_+}\left(1+\cdots\right) \tag{4.44}$$

for the exponents

$$\begin{aligned} \beta_+ &= \frac{1}{2}\left(1+\sqrt{1+m^2\ell^2}\right) := \frac{\Delta}{2}\,, \\ \beta_- &= \frac{1}{2}\left(1-\sqrt{1+m^2\ell^2}\right) = \frac{d-\Delta}{2}\,. \end{aligned} \tag{4.45}$$

We usually refer to the coefficient $A(k)$ as the source, while $B(k)$ is the expectation value. We shall see in more detail why this is the case in the chapter on holographic renormalization. In order to define the mode function $f_k(z)$ we should work with the normalized version, that is

$$f_k(z) \sim (1-z)^{\frac{d-\Delta}{2}}\left(1+\cdots\right) + \frac{B(k)}{A(k)}(1-z)^{\frac{\Delta}{2}}\left(1+\cdots\right)\,. \tag{4.46}$$

From this we can finally calculate the flux factor

$$\mathcal{F}(k) = d\frac{B(k)}{A(k)}\,. \tag{4.47}$$

We have, for simplicity, taken the coefficients A and B to be real, but a similar expression is obtained if they are allowed to be complex. We see that, morally speaking, the flux factor and thus the Green function are given as

$$G = \frac{\text{expectation value}}{\text{source}}\,. \tag{4.48}$$

This is a general feature of holographic two-pt functions, however our arguments here are not sufficient to establish this result in any generality. We shall return to the detailed derivation below in the chapter on holographic renormalization. However, A and B are not determined from the asymptotic analysis. As we have seen above, in order to do so, we need to impose the infrared boundary condition. This is of course exactly what the full solution - the hypergeometric function - achieves.

The Retarded Green Function

The most general expression for the correlation function is not particularly enlightening. Let us specialize to $h_L = h_R = 1$, that is $\Delta = 2, s = 0$. Then the retarded Green function can be written in terms of the digamma function

$$\psi(x) := \frac{\Gamma'(x)}{\Gamma(x)} \tag{4.49}$$

as

$$G_R(p_+, p_-) = \frac{K}{\pi^2} \frac{p_+ p_-}{\ell T_L T_R} \left[\psi\left(1 - \frac{ip_+}{2\pi T_L}\right) + \psi\left(1 - \frac{ip_-}{2\pi T_R}\right)\right] . \tag{4.50}$$

Here we have defined the left and right moving temperatures

$$T_R = \frac{\rho_+ + \rho_-}{2\pi} \qquad T_L = \frac{\rho_+ - \rho_-}{2\pi} \tag{4.51}$$

and

$$\begin{aligned} p_+ &= \pi \ell T_L (k_+ + k_-) = \frac{\omega - k}{2} \, , \\ p_- &= \pi \ell T_R (k_+ - k_-) = \frac{\omega + k}{2} \, . \end{aligned} \tag{4.52}$$

Having obtained the full expression for the retarded Green function (4.50), let us now discuss some of its properties. It follows from the analytical properties of the dilogarithm (4.49) that this expression has simple poles (and only simple poles) at the discrete frequencies

$$\begin{aligned} \omega_n^{(L)} &= k - i4\pi T_L(h_L + n) \qquad n \in \mathbb{Z}_{\geq 0} \\ \omega_n^{(R)} &= -k - i4\pi T_R(h_R + n) \qquad n \in \mathbb{Z}_{\geq 0} \,. \end{aligned} \tag{4.53}$$

Strictly speaking we only established this result for $h_L = h_R = 1$, but with some more work it can be shown to be true as quoted for the more general case as well. This series of pole frequencies descending into the lower half-plane is typical of thermal correlation functions in holography. The details, of course, vary from operator to operator and dimension to dimension.

Comments

1. The mode function $f_k(k_+, k_-, z) := f_k(z)$ behaves near the boundary as

$$f_k(z) \sim A(\omega, k)(1 - z)^\beta + \text{subleading} \tag{4.54}$$

   The origin of the poles (4.53) can be traced back to the vanishing of $A(\omega)$ at the discrete frequencies $\omega_{(n)}^{(L,R)}$ above. In other words the poles in the retarded correlation function correspond to special mode solutions where the coefficient of the leading power near the boundary vanishes. In holographic parlance[7] one calls such modes 'normalizable'. The corresponding mode solutions are called quasinormal modes which arise in gravity quite generally as the ring-down frequencies of black holes. Side remark: the ring-down pattern observed in LIGO's gravitational wave signal is due to such quasinormal modes and occurs at the charactersitic quasinormal frequencies of the asymptotically flat Kerr black hole which can be determined in analogous fashion to our BTZ calculation.

2. As we already touched upon in the previous item, these frequencies and their corresponding modes are tied to the relaxation of perturbations around black

[7]Not without good reason: It is exactly for such modes that a finite energy integral can be defined.

hole solutions. We can define a decay time

$$\left.\frac{1}{\tau_{\text{rel.}}}\right|_{\text{QFT}} = \text{Im}(\omega^*) \tag{4.55}$$

as the imaginary part of the leading quasinormal mode ω^*. This is defined to be the mode nearest to the real axis. In the example above, Eq. (4.53), for any momentum k, this is the $n = 0$ mode. Intuitively we understand this as follows: the solution at late time can be expressed as a sum over quasinormal modes which all decay exponentially. It is then clear that the sum will be dominated by the term with the slowest decay,

$$\phi(t) \sim \sum_n \mathcal{A}_n e^{-i\omega_n t} \longrightarrow \mathcal{A}_0 e^{-\text{Im}(\omega_0)t} e^{-i\text{Re}(\omega_0)t} \,. \tag{4.56}$$

This means that the characteristic time scale for perturbations to decay to thermal equilibrium is that defined in (4.55). In the field theory these show up as poles in the retarded correlation function, as explained above. In fact there is a rich connection between this story and the hydrodynamic description of quantum field theories. We will have occasion to return to this point later on.

3. Notice that all quasinormal modes of the retarded correlation function have frequencies located in the lower half plane. We can now see that this is a consequence of the stability of the state we compute the correlation function in. Holographically, stability of the state is the same as stability of the background solution. If we had a mode in the upper half plane, then by the same argument as the one leading to Eq. (4.56) we would deduce that there is an exponentially growing solution. An exactly analogous line of thought leads to the conclusion that the advanced propagator has only poles in the upper half complex plane.

In the light of the above comments let us end this section by summing up that poles in retarded correlation functions are in on-to-one correspondence with quasi-normal modes. Furthermore there is a precise quantitative connection between these quasinormal frequencies and the thermalization of dual QFT. We explored this relation in the context of the specific example of a two-dimensional CFT in a thermal state, but it is in fact true very generally. Schematically, for a field in an asymptotically AdS_{d+1} spacetime, one can write the mode function asymptotically near the boundary

$$f \sim Az^{\Delta} + Bz^{d-\Delta} + \cdots \tag{4.57}$$

where A is the non-normalizable mode and B the normalizable one. The retarded Green function is then given, equally schematically as

$$G_R \sim \frac{B}{A} + \text{contact terms}$$

It is then clear that the frequencies for which A vanishes show up as poles. These are of course precisely the quasinormal modes (often denoted QNM).

Bibliography

[1] G. W. Gibbons, S. W. Hawking and M. J. Perry, "*Path integrals and the indefiniteness of the gravitational action*" Nuclear Physics B (1978) 1, doi: http://dx.doi.org/10.1016/0550-3213(78)90161-X

[2] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, "*Gauge Theory Correlators from Noncritical String Theory*," Phys. Lett. B **428** (1998) 105 doi:10.1016/S0370-2693(98)00377-3 [hep-th/9802109].

[3] E. Witten, "*Anti-de Sitter Space and Holography*," Adv. Theor. Math. Phys. **2** (1998) 253 [hep-th/9802150].

[4] J. Erlich, E. Katz, D. T. Son and M. A. Stephanov, "*QCD and a Holographic Model of Hadrons*," Phys. Rev. Lett. **95** (2005) 261602 doi:10.1103/PhysRevLett.95.261602 [hep-ph/0501128].

[5] D. T. Son and A. O. Starinets, "*Minkowski Space Correlators in AdS / CFT Correspondence: Recipe and Applications*," JHEP **0209** (2002) 042 doi:10.1088/1126-6708/2002/09/042 [hep-th/0205051].

[6] C. P. Herzog and D. T. Son, "*Schwinger-Keldysh Propagators from AdS/CFT Correspondence*," JHEP **0303** (2003) 046 doi:10.1088/1126-6708/2003/03/046 [hep-th/0212072].

Chapter 5

Nonequilibrium Physics

5.1 Nonequilibrium Physics: The Field Theory Perspective

We are all familiar with the beautiful connection between Euclidean Quantum Field Theory and equilibrium Statistic Mechanics[1]. However, many physical phenomena of interest do not happen in equilibrium. Examples are plentiful and reach from cosmology, heavy ion physics and cold atoms to the physics of Life. All of these, and presumably many others, highlight the fundamental importance of understanding quantum systems out of equilibrium. This leads quite naturally to the study of thermalization, isotropization and rather generally the process of relaxation towards an equilibrium state for a system that finds itself initially away from it. In order to appreciate where the difference lie with the usual sitaution, let us review first the familiar case, and in particular by looking at a two-point correlation function. Let us look at a time-ordered correlation function of an operator $\psi(x^\mu) = \psi(t, \vec{x})$ in a general state, described by a density matrix ρ. We have

$$\begin{aligned} iG(x^\mu, x'^\mu) &= \langle T \left\{ \psi(x^\mu)\psi^\dagger(x'^\mu)) \right\} \rangle \\ &= \mathrm{Tr}\rho\, T \left\{ \psi(x^\mu)\psi^\dagger(x'^\mu) \right\}, \end{aligned} \tag{5.1}$$

[1]This chapter draws from various sources, among them [1, 2, 3, 4].

where ρ is a general, not necessarily equilibrium, density matrix. That is to say that we allow for a Hamiltonian $\mathcal{H}(t)$ which explicitly depends on time. The operators appearing in the trace are Heisenberg picture objects. We will often suppress the Lorentz indices on spacetime events writing $x^\mu \leftrightarrow x$ in order to de-clutter our formulae.

5.1.1 Equilibrium

Before moving on to the general case, let us review briefly how to proceed along the lines usually encountered in quantum field theory textbooks, and let us highlight why this formalism is intimately tied to equilibrium. Let us consider the case of a pure state $|\Psi_0\rangle$, usually taken as the ground state of some free Hamiltonian H_0, so that $\rho = |\Psi_0\rangle\langle\Psi_0|$. Then the full time-ordered correlation function is obtained from

$$\begin{aligned} iG(x,x') &= \langle\Psi_0|T\left\{\psi(x)\psi^\dagger(x')\right\}|\Psi_0\rangle \\ &= \frac{\langle\Psi_0|T\left\{S(\infty,-\infty)\hat{\psi}(x)\hat{\psi}(x')\right\}|\Psi_0\rangle}{\langle\Psi_0|S(\infty,-\infty)|\Psi_0\rangle}\,. \end{aligned} \tag{5.2}$$

Here we have introduced the interaction picture field operator $\hat{\psi}(x)$ with respect to the splitting of the Hamiltonian into free and interacting part, $H = H_0 + V(t)$. We have the S-matrix

$$S(\infty,-\infty) = T\exp\left(-i\int_{-\infty}^{\infty} dt_1 \hat{V}(t_1)\right)\,. \tag{5.3}$$

The correlation function $G(x,x')$ is then calculated using Wick's theorem, referring to the non-interacting ground state $|\Psi_0\rangle$. This gives rise to the usual Feynman-Dyson perturbation theory familiar from standard quantum field theory textbooks. Of course this expression is valid in Lorentzian time, so one may ask oneself why this expression should not also apply in the non-equilibrium setting. In fact in order to arrive at (5.2) and the resulting Feynman-Dyson perturbation theory we have made a number of hidden assumptions, which shall turn out not to be valid generally. We shall now attempt to derive this expression in the non-equilibrium context, see how it fails, and what it gets replaced by.

5.1.2 Nonequilibrium

Let us consider a general density matrix

$$\rho = \sum_{\Phi} \rho_{\Phi} |\Phi\rangle\langle\Phi| \,, \tag{5.4}$$

where we have in mind a set of coefficients ρ_Φ that do not lead to an equilibrium state, as would for example be the case if we took $\rho_\Psi \sim e^{-\beta E_\Phi}$. It is evident that we may consider one state at a time and assemble the full answer in the end as a sum over the individual results. We shall thus consider one particular term,

$$iG_\Phi(x,x') = \langle\Phi| T\left\{\psi(x)\psi^\dagger(x')\right\}|\Phi\rangle \,. \tag{5.5}$$

Let us decompose the Hamiltonian into $\mathcal{H}(t) = H + H'(t)$, where H is the equilibrium part and $H'(t)$ is the time-dependent part. We now consider an 'interaction' picture, but with respect to the decomposition into H and $H'(t)$. The hat on the operator $\hat{\psi}(x^\mu)$ denotes again the interaction picture with respect to that decomposition ,

$$\hat{\psi}(x^\mu) = e^{iHt}\psi(0,\vec{x})e^{-iHt} \tag{5.6}$$

and the S-matrix is given by

$$S(t,t') = T\,\exp\left(-i\int_{t'}^{t} dt_1 \hat{H}'(t_1)\right), \tag{5.7}$$

where also the time-dependent part of the Hamiltonian should be evaluated in the interaction picture,

$$\hat{H}'(t) = e^{iHt}H(t)e^{-iHt} \,. \tag{5.8}$$

We can now put these expressions together to rewrite the correlation function as the expectation value

$$iG_\Phi(x,x') = \langle\Phi|T\left\{S(0,t)\hat{\psi}(t,\vec{x})S(t,0)S(0,t')\hat{\psi}^\dagger(t',\vec{x}')S(t',0)\right\}|\Phi\rangle \,. \tag{5.9}$$

Here we have used the relation between the time evolution in the Heisenberg picture and the interaction picture

$$\begin{aligned}\psi(\vec{x},t) &= \underset{\text{Heisenberg picture}}{U(0,t)\psi(\vec{x})U(t,0)} \\ &= \underset{\text{interaction picture}}{S(0,t)\hat{\psi}(t,\vec{x})S(t,0).}\end{aligned} \tag{5.10}$$

We then write

$$|\Phi\rangle = S(0,\pm\infty)|\Phi(\pm\infty)\rangle_I \tag{5.11}$$

where $|\Phi\rangle$ denotes the interaction picture state at $t=0$ (or equivalently the Heisenberg picture state at $t=0$, since we take $t=0$ as the time where the two coincide), and $|\Phi(\pm\infty)\rangle_I$ denotes the interaction picture state at $t=\pm\infty$. Thus, the correlator (5.5) can be rewritten as

$$\begin{aligned}iG_\Phi &= {}_I\langle\Phi(\infty)|S(\infty,0)\,T\left\{S(0,t)\hat{\psi}(t,\vec{x})S(t,t')\hat{\psi}^\dagger(t',\vec{x}')S(t',0)\right\}\,S(0,-\infty)|\Phi(-\infty)\rangle_I \\ &= {}_I\langle\Phi(\infty)|T\left\{S(\infty,t)\,\hat{\psi}(t,\vec{x})S(t,t')\hat{\psi}^\dagger(t',\vec{x}')S(t',-\infty)\right\}|\Phi(-\infty)\rangle_I \\ &= {}_I\langle\Phi(\infty)|T\left\{S(\infty,-\infty)\,\hat{\psi}(t,\vec{x})\hat{\psi}^\dagger(t',\vec{x}')\right\}|\Phi(-\infty)\rangle_I.\end{aligned} \tag{5.12}$$

Note that $S(\infty,0)$ and $S(0,-\infty)$ can be pulled inside the time ordering operator as shown, because they are themselves time ordered. In the last line we have used the time-ordering operator to write these operations in a more compact form. In order to compute such expectation values, we evidently need to know about the overlap of states obtained by acting with field operators on $|\Phi(-\infty)\rangle_I$ and $|\Phi(\infty)\rangle_I$. Typically these are not the same, so some way of computing the overlap has to be developed. Very often one does not explicitly bother with with this complication, instead one works directly with expressions of the form (5.2). We recover this form of the expansion by assuming that

$$|\Phi(\infty)\rangle_I = e^{iL}\Phi|(-\infty)\rangle_I \qquad \text{(assumption)}, \tag{5.13}$$

which states that the two states sandwiching the expression (5.12) are the same, up to a phase. Under this assumption we can determine this phase as

$$e^{iL} = {}_I\langle\Phi(-\infty)|S(\infty,-\infty)|\Phi(-\infty)\rangle_I, \tag{5.14}$$

and so (5.12) becomes equivalent to the usual Feynman-Dyson expression (5.2). But is this assumption reasonable and justified? In general the answer is 'no', although in many applications, for example when one computes typical particle-physics cross sections, it is. The usual justification for the required steps is adiabaticity. For example one constructs the interacting ground state, by having a time-dependent Hamiltonian which 'switches on' the interactions infinitely slowly, and switches them off again equally slowly

$$\mathcal{H}_\epsilon = H + e^{-\epsilon|t|}H'\,. \tag{5.15}$$

In other words the state $|\Phi_0\rangle$, i.e. the ground state of $\mathcal{H} = H + H'$ is constructed by the limiting procedure

$$|\Phi(\pm\infty)\rangle_I = \lim_{\epsilon\to 0} S(\pm\infty, 0)|\Phi_0\rangle, \tag{5.16}$$

While involving explicit time dependence, this construction assumes quantum adiabatic evolution, that is that for sufficiently slow evolution the state at any given time is the instantaneous ground state of the Hamiltonian. Since we very slowly morph one Hamiltonian into the other, in this manner one 'drags' one ground state into the other. In this case the assumption (5.13) is indeed justified. However, the adiabatic theorem does not hold away from equilibrium and so this construction is not in general justified for a Hamiltonian of the form $\mathcal{H}(t) = H + H'(t)$. To sum up: for a general non-equilibrium evolution the final state is not known, and in particular it is not generally going to be the same as the initial state (up to a phase rotation). It is, in some sense, the goal of nonequilibrium calculations to determine the final state. It is therefore necessary to develop a formalism that does not refer to a final state that is assumed known a priori.

How should one proceed in this more general situation? The answer lies in the expressions we already derived. Let us refer back to $|\Phi(-\infty)\rangle_I$ which we understand. We can obtain the state at ∞ by using the S matrix to evolve in time all the way to $-\infty$. We thus write

$$|\Phi(\infty)\rangle_I = S(\infty, -\infty)|\Phi(-\infty)\rangle_I \tag{5.17}$$

So now the correlation function is

$$iG_\Phi(x, x') = {}_I\langle\Phi(-\infty)|S(-\infty, \infty)T\left\{S(\infty, 0)\,\psi(x)\psi^\dagger(x')S(0, -\infty)\right\}|\Phi(-\infty)\rangle_I\,. \tag{5.18}$$

This expression is the basis of what is often referred to as the Schwinger-Keldysh formalism. The first observation is an æsthetic one: The expression is no longer time ordered. Reading from right to left, we first evolve from $-\infty$ to ∞ with two operator insertions on the way, but then back again from ∞ to $-\infty$. In fact, we can think of this as time ordered, but along a new, doubled contour. This is usually called the Schwinger-Keldysh contour, denoted as $\gamma = \gamma_+ \oplus \gamma_-$. The contour reaches from $-\infty$ to ∞ along γ_+ and the loops back to $-\infty$ along γ_-. It will be convenient to describe this with a continuous 'contour time' s; forward evolution in s coincides with forward evolution along γ_+, but corresponds to backward evolution in t along γ_-. Eq. (5.18) can then be thought of as contour-ordered:

$$iG_\Phi(x, x') = {}_I\langle\Phi(-\infty)|T_c\left\{S(-\infty, -\infty)_c\,\psi(x)\psi^\dagger(x')\right\}|\Phi(-\infty)\rangle_I \tag{5.19}$$

with the double-time S matrix:

$$S_c \;=\; T_c \exp\left(-i\oint_{\gamma=\gamma_+\oplus\gamma_-} ds\,\hat{H}'(s)\right) \tag{5.20}$$

Under contour order T_c we consider any time on γ_+ to be *earlier* than any time on γ_-.

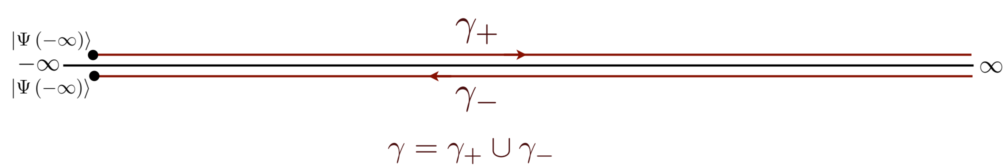
Comments

We conclude this section with a few comments on the structure and frequently encountered variants and extensions of this formalism. In the next section we will turn to a very important application, namely that of linear response theory.

1. A useful way to look at the expression (5.19) is achieved by rewriting it as

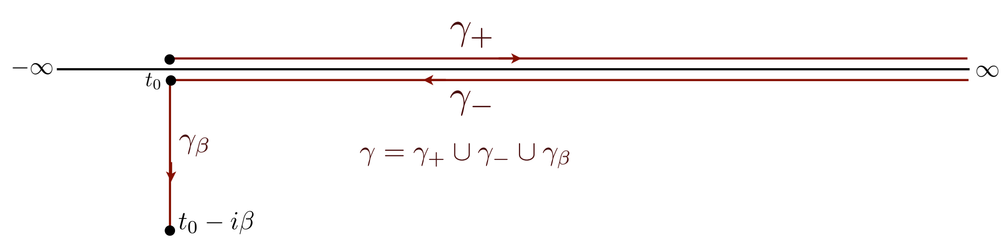
$$iG_\Phi = \mathrm{Tr}\left(\rho(-\infty)T_c\left\{S_c(-\infty,\infty)\hat{\psi}(x)\,\hat{\psi}^\dagger(x')\right\}\right), \tag{5.21}$$

   where the density matrix $\rho(-\infty) = |\Phi(-\infty)\rangle\langle\Phi(-\infty)|$ is known by construction.



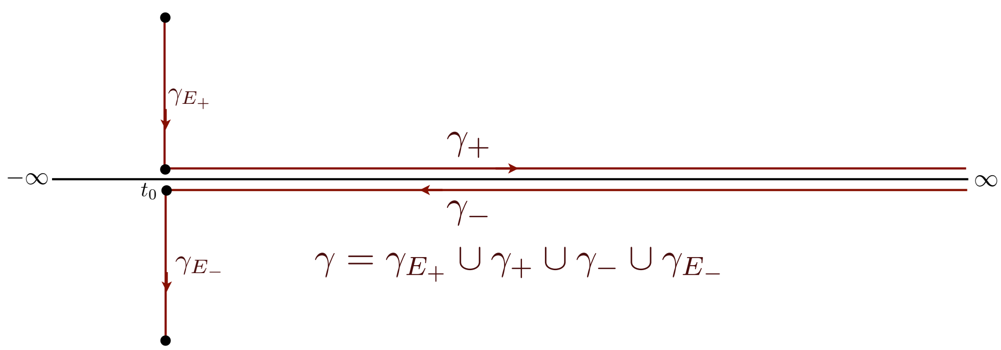
   In fact we could start from any known $\rho = |\Phi_0\rangle\langle\Phi_0|$ at some time t_0, earlier than any operator insertions in the problem, although in practice any examples other than equilibrium are difficult to to construct, especially in interacting theories.

2. In the latter scenario it can be very useful to construct the initial state or density matrix via Euclidean evolution. This leads to yet another part added to the contour, this time in the Euclidean time direction. For the case of the thermal ensemble the Euclidean section runs from $t_0 - i\beta$ to t_0. One can include this in the contour order prescription by defining any time on the Euclidean section to be later than any of the two Lorentzian sections. It is also useful to note that any Hamiltonian operator can be used on the Euclidean section, and that this choice essentially determines the state (more on this below). In the condensed-matter literature this is sometimes called the Kadanoff-Baym, or Konstantinov Perel', contour.



Essentially the idea is that the Euclidean evolution can produce a certain state, in the present case the thermal equilibrium, which is then used as the basis for the Lorentzian evolution. We will postpone a more intuitive discussion to Section 5.4

3. The same idea can be pushed further, using the Euclidean evolution in order to prepare non-trivial intital states, which do not necessarily have to be the thermal ensemble for some Hamiltonian H. This is the so-called 'in-in' prescription.



We should think as the γ_{E_+} part as preparing the ket $|\Psi\rangle$, while the γ_{E_+} prepares the bra $\langle\Psi|$ and the integral over the entire contour γ produces the expectation value

$$\langle\Psi|\mathcal{O}(t)\cdots|\Psi\rangle \,.$$

The ellipsis indicates that we may insert any number of operators on the Lorentzian part of the contour, whose Lorentzian time evolution in the state $|\Psi\rangle$ we would like to study.

4. The way we have derived the formalism it was most natural to have operator insertions only on the γ_+ part of the contour. However, operators can be inserted on γ_+ and γ_-, and in some problems it is natural to insert them on both parts of the contour. Actually, had we inserted the backwards evolution in Eq. (5.18) to the right of the time-ordered expression, by inverting the relation (5.17), the operators would naturally have been inserted on the γ_- part of the resulting contour. In fact, from the holographic perspective, we have already seen that in order to calculate the full set of Lorentzian n-point functions we were naturally led to insert sources on each side, which here

are reinterpreted as the two parts of the contour. We shall revisit this in the next section, when we study fluctuation-dissipation relations.

Unfortunately all this formalism leads to a complicated diagrammatic expansion involving three different propagators, instead of just the one of the more familiar equilibrium theory.

5. The two-time approach is closely tied to perturbation theory, in the same sense that the usual Feynman-Dyson approach is. It is thus in serious trouble for strongly coupled systems and it is here where holography can play a decisive role in elucidating strongly coupled non-equilibrium physics. Furthermore the resulting time dependent perturbation theory can lead to secular divergences even at weak coupling, further complicating the situation.

Contour Heisenberg Picture

[to be completed...]

5.2 Kubo Formulae

5.2.1 Linear Response Theory

We will now describe a very common application of the ideas presented above. Consider perturbing the system by a time-dependent source. A classical example would be an electric field that is switched on - perhaps with a ramp - at some time t_0 and switched off again at a later time[2]. What is the time evolution of physical quantities in this scenario? The physical question one may want to ask in this context is that of transport of charge, or heat or similar, as quantified by various different conductivities. The mathematical problem is clearly that of nonequilbrium evolution, and in this case of operators in the perturbed field theory.

[2] The transport problem can be studied without introducing the full formalism of nonequilbrium field theory, but we still think it is useful to see how this fits into the very general framework of Schwinger-Keldysh. In some sense the SK point of view is the most complete.

We therefore consider a time dependent Hamiltonian of the following form

$$\mathcal{H}(t) = H + H' = H + \int d^n x \, F_i(t, \vec{x}) \, \mathcal{O}_i(\vec{x}) \,, \tag{5.22}$$

where F_i is the source for the operator $\mathcal{O}_i$, which is switched on at some time t_0, so $F(t, \vec{x}) = 0$ for $t < t_0$. This is a special case of a time dependent perturbation, one that linearly couples one of the operators in our theory to an external source. Working through the formalism above, to linear order in $F_i(t, \vec{x})$ is what gives rise to linear response theory and ultimately to what is referred to as a Kubo Formula. Suppose now we would like to follow the time evolution of some (possibly) different operator $\mathcal{Q}(\vec{x})$ in time:

$$\langle \mathcal{Q}(t) \rangle = \mathrm{Tr}\left[\rho(t_0) T_c \left\{ S_c(t_0, t_0) \mathcal{Q}(t, \vec{x}) \right\}\right] \tag{5.23}$$

We now disentangle again the contour ordering, keeping in mind that it turns into anti-time ordering on the backward contour, to obtain

$$\left\langle \bar{T} \exp\left(i \int_{t_0}^{t} H'(s) ds \right) \mathcal{Q}(t) \, T \exp\left(-i \int_{t_0}^{t} H'(s) ds \right) \right\rangle_{\rho_0} \,, \tag{5.24}$$

evaluated, as indicated, with respect to the initial density matrix ρ_0. We now expand this expression to first order in the source (linear response)

$$\begin{aligned} \langle \mathcal{Q}(t) \rangle &= \langle \mathcal{Q}(t) \rangle_{\rho_0} + i \int_{t_0}^{t} ds \, d^n \vec{x} \, \langle [H'(s), \mathcal{Q}(t)] \rangle_{\rho_0} + \cdots \\ &= \int_{t_0}^{\infty} ds \, d^n \vec{x} F_i(s, \vec{x}) i \Theta(t - s) \langle [\mathcal{O}_i(s), \mathcal{Q}(t)] \rangle_{\rho_0} \\ &= \int_{-\infty}^{\infty} ds \, d^n \vec{x} F_i(s, \vec{x}) G^R_{\mathcal{O}_i Q}(t - s, |\vec{x} - \vec{y}|), \end{aligned} \tag{5.25}$$

where we have assumed that the equilibrium average of $\mathcal{Q}$ vanishes and we introduced a step function in order to extend the integration range. In the last line we have further extended the integration range to minus infinity, utilizing the vanishing of the source for times earlier than t_0. The resulting integrand is then the retarded Green function as shown. The last line of Eq. (5.25) is the basic relation of linear response theory upon which one can develop a whole edifice of linear transport theory. We shall not develop this in all of its details, but rather contend ourselves with a classical example, namely that of the Kubo formula for charge transport. We refer the interested reader to the literature for more information (e.g. [5]).

5.2.2 Example: Electrical Conductivity

The electrical conductivity, within the purview of linear response theory, is the resulting current in response to a an applied electric field, to linear order,

$$\left\langle J_i(\omega, \vec{k})\right\rangle = \sigma_{ij}(\vec{k},\omega)E_j(\vec{k},\omega)\,. \tag{5.26}$$

By considering the contribution from bound currents and free charge carriers, one deduces the conductivity matrix

$$\sigma_{ij} = \frac{-e\rho}{i\omega}\delta_{ij} + \frac{\chi_{ij}}{i\omega}\,. \tag{5.27}$$

Where ρ is the static charge density and χ_{ij} is the so-called charge susceptibility, given by a correlation function along the lines above, as

$$\chi_{ij}(\omega, \vec{k}) = -\int_{-\infty}^{\infty} dt d^3\vec{x}\Theta(t)e^{i(\omega t-\vec{k}\cdot\vec{x})}\left\langle [J_i(t,\vec{x}), J_j(0,0)]\right\rangle\,. \tag{5.28}$$

It is the latter expression which is usually referred to as the Kubo formula for the charge conductivity. It is conventional to refer to the $\omega \to$ as the DC conductivity while the finite frequency part gets the name optical conductivity. One of the main advantages of the linear response framework is that we are able to calculate nonequilibrium notions, such as transport coefficients of charge etc. in terms of equilibrium correlation functions. Note that this assumes, as is often reasonable, that ρ_0 is indeed an equilibrium density matrix. In a sense this is possible because we only consider small departures from equilibrium.

Let us now discuss a set of powerful relations between correlation functions that follow from the general Schwinger-Keldysh setup, when considering such near-equilibrium problems. These relations go under the general name of fluctuation dissipation relations.

5.3 KMS States and Fluctuation Dissipation Relations

We shall now derive a set of boundary conditions for correlation functions on the Kadanoff-Baym contour, known as the Kubo-Martin-Schwinger (KMS) conditions. These are a property of time-dependent expectation values with an equilibrium initial density matrix. In a precise sense these conditions characterise thermality of correlation functions. We will then use them to prove the so-called fluctuation dissipation theorem on two-point functions.

In this section we will first present the general statement for n-point functions. However, our main interest is in two-point functions, and the reader not interested in the technicalities of the general case can safely jump ahead to Section 5.3.2, where the two-point function case presented in a self-contained manner.

5.3.1 KMS Condition: General Case

Let us start with a general contour ordered correlation function

$$G_n\left(1\dots n, 1'\cdots n'\right) = \mathrm{Tr}\left[e^{-\beta H}T_c\left\{\psi(t_1,\vec{x}_1)\dots\psi(t_n,\vec{x}_n)\psi^\dagger(t'_1,\vec{x}'_1)\dots\psi(t'_n,\vec{x}'_n)\right\}\right] \quad (5.29)$$

The notation here means that the number argument of the correlation function stands in for the tuple of time and position $(t,\vec{x})$ of the operator insertion, so e.g. $1 \leftrightarrow (t_1,\vec{x}_1)$ and so on. We have used contour ordering but have explicitly displayed the Euclidean evolution operator in the left-most position (so that the full expression is still contour ordered along the Kadanoff-Baym contour). This will make our subsequent manipulations more transparent. Let us also suppose, for simplicity, that all operators are bosonic, which means that operators commute[3] inside the T_c operator. Now let us suppose that the time of one of the ψ insertions, say the k^{th} one, is actually $t_k = t_0$, i.e. we have the operator $\hat{\psi}(k) \leftrightarrow \hat{\psi}(t_0, x_k)$ inserted somewhere in the correlation function. Since t_0 is the earliest time on the contour the T_c operation puts this automatically to the very right. Cyclicity of

[3] The only difference with fermions is that they anticommute, which means that we would have to keep track of a lot of signs at various intermediate steps.

the trace then permits us to perform the following operations on the correlation function $G\,(1\cdots(t_0,\vec{x}_k)\dots n, 1'\dots n')$:

$$
\begin{aligned}
& \mathrm{Tr}\left[e^{-\beta H} T_c\left\{\psi(t_1,\vec{x}_1)\dots\psi(t_n,\vec{x}_n)\psi^\dagger(t_1',\vec{x}_1')\dots\psi(t_n',\vec{x}_n')\right\}\right] \\
= \quad & \mathrm{Tr}\left[\hat{\psi}(t_0,\vec{x}_k)e^{-\beta H} T_c\left\{\psi(t_1,\vec{x}_1)\dots\psi(t_n,\vec{x}_n)\psi^\dagger(t_1',\vec{x}_1')\dots\psi(t_n',\vec{x}_n')\right\}\right] \\
= \quad & \mathrm{Tr}\left[e^{-\beta H}\hat{\psi}(t_0-i\beta,\vec{x}_k) T_c\left\{\psi(t_1,\vec{x}_1)\dots\psi(t_n,\vec{x}_n)\psi^\dagger(t_1',\vec{x}_1')\dots\psi(t_n',\vec{x}_n')\right\}\right] \\
= \quad & G(1\dots(t_0-i\beta,\vec{x}_k),\dots n, 1'\dots n')\,. \qquad (5.30)
\end{aligned}
$$

The final expression follows because in the previous line we may pull the $\psi(t_0 - i\beta,\vec{x}_k)$ operator back into the T_c ordering as it is now inserted at the latest time on the contour and all operators inside the T_c sign commute, so we may then further push it back into its original position. What this has shown us is that the appropriate boundary condition on the contour is for each correlation function with bosonic insertions to be periodic under translation along the entire Kadanoff-Baym contour. A sign may be picked up for a correlation function of fermions, leading to anti-periodic boundary conditions. In simpler and more familiar terms, the (bosonic) correlation function is periodic with respect to translation in imaginary time with period β. A similar relation holds for each operator insertion, including the daggered ones. These relations can be used as the defining properties of a thermal state, since they are a consequence of the equilibrium density matrix $e^{-\beta H}$ inserted on the Euclidean part of the contour.

The above treatment was rather abstract, so let us now see what these relations are actually good for. To this end, let us focus on the case of the two point function.

5.3.2 KMS Condition: Two-point Function

It is useful to define the so called 'greater' and 'lesser' two point function, distinguished by the ordering of operators inside the trace,

$$\begin{aligned} iG^>(t,t') &= \mathrm{Tr}\left[e^{-\beta H}\hat{\psi}(t)\hat{\psi}^\dagger(t')\right] , \\ iG^<(t,t') &= \mathrm{Tr}\left[e^{-\beta H}\hat{\psi}^\dagger(t')\hat{\psi}(t)\right] . \end{aligned} \tag{5.31}$$

Here we have suppressed the spatial argument of each operator. These are related to the more familiar retarded and advanced correlators as

$$\begin{aligned} G^R(t,t') &= \Theta(t-t')\left(G^>(t,t')-G^<(t,t')\right) \\ G^A(t,t') &= \Theta(t'-t)\left(G^<(t,t')-G^>(t,t')\right) , \end{aligned} \tag{5.32}$$

but as we shall see have a more direct relation with the two-time contour formalism than the latter. It is also useful to define a third kind of two-point function

$$G^K(t,t') = G^>(t,t') + G^<(t,t') , \tag{5.33}$$

where the subscript stands for 'Keldysh'. It is also often referred to as the 'symmetric' two-point function for obvious reasons.

As a special case of the general KMS relation we can derive a condition on the two-point functions

$$\begin{aligned} iG^>(t,t') &= \mathrm{Tr}\left[e^{-\beta H}\hat{\psi}(t)e^{\beta H}e^{-\beta H}\hat{\psi}^\dagger(t')\right] \\ &= \mathrm{Tr}\left[e^{-\beta H}\hat{\psi}^\dagger(t')\hat{\psi}(t+i\beta)\right] \\ &= iG^<(t+i\beta,t') \end{aligned} \tag{5.34}$$

with an analogous statement for on the second argument. So in summary the thermal Green functions are periodic in imaginary time

$$\begin{aligned} iG^>(t,t') &= iG^<(t+i\beta,t') \\ iG^>(t,t') &= iG^<(t,t'-i\beta) \end{aligned} \tag{5.35}$$

with period β (up to exchanging the 'greater' with the 'lesser' Green function). The second relation is derived in an analogous fashion, or simply follows from time-translation invariance.

5.3.3 Fluctuation-Dissipation Theorem

Despite its humble provenance as a mere boundary condition the KMS relation is extremely useful in practice. It implies a relation between the three different correlation functions defined above, most conveniently written in frequency space

$$G^R(\omega) - G^A(\omega) = \tanh\frac{\beta\omega}{2}G^K(\omega). \tag{5.36}$$

This relation is known as the fluctuation-dissipation theorem (FDT). We will first derive it and then make a few comments on its physical meaning. To derive this relation let us consider the Fourier transform of the Keldysh Green function. By time translation invariance $G^<(t,t') = G^<(t-t')$, and so

$$\begin{aligned} G^K(\omega) &= \int_{-\infty}^{\infty} \left(G^>(t) + G^<(t)\right) e^{-i\omega t} dt \\ &= \int_{-\infty}^{\infty} \left(G^>(t) + G^>(t+i\beta)\right) e^{-i\omega t} dt \\ &= (1+e^{-\beta\omega})S(\omega)\,, \end{aligned} \tag{5.37}$$

where $S(\omega)$ is the Fourier transform of the 'greater' Green function $G^>(t-t')$. By a similar application of the KMS condition we have

$$G^A(\omega) - G^R(\omega) = (e^{-\beta\omega}-1)S(\omega)\,, \tag{5.38}$$

using the fact that

$$G^R(t) - G^A(t) = G^>(t) - G^<(t)\,. \tag{5.39}$$

This last equation follows from the definition, Eq. (5.32) and the fact that $\Theta(t) + \Theta(-t) = 1$. By combining the last two equations, the FDT (5.36) follows immediately.

Comments

1. Firstly, let us connect back to what we learnt about Lorenztian Green functions in the holographic context. Referring to the contour definition of correlators, we have

$$\begin{aligned} iG_{12}(t,t') &= \langle \psi_1(t)\psi_2^\dagger(t')\rangle \\ iG_{21}(t,t') &= \langle \psi_2(t)\psi_1^\dagger(t')\rangle \\ iG_{11}(t,t') &= \langle T\psi_1(t)\psi_1^\dagger(t')\rangle \\ iG_{22}(t,t') &= \langle \bar{T}\psi_2(t)\psi_1^\dagger(t')\rangle \end{aligned} \tag{5.40}$$

where the notation $\psi_1(t)$ means the insertion of the operator $\psi(t)$ on the forward branch of the contour at time t, while $\psi_2(t)$ stands for the insertion on the backward branch at the same time t. We have so far avoided making explicit reference to the part of the contour concerned because it is complicated and in fact not necessary if one uses contour time ordering as we have been doing throughout. We reintroduce this notation here as it is necessary to make contact with our previous holographic results. From the above Eq. (5.40), and using our previous definitions (5.32) it follows that

$$\begin{aligned} G_{11} &= \frac{1}{2}\left(G^A + G^R + G^K\right) \\ G_{12} &= \frac{1}{2}\left(G^A - G^R + G^K\right) \\ G_{21} &= \frac{1}{2}\left(G^R - G^A + G^K\right) \\ G_{22} &= \frac{1}{2}\left(-G^R - G^A + G^K\right) \end{aligned} \tag{5.41}$$

where for simplicity of notation we have suppressed all arguments. To be explicit, all Green functions here are evaluated at the same points, e.g. $G^R(t,t')$. We now use that $G^R(t,t')^\dagger = G^A(t,t')$ together with the fluctuation dissipation theorem to find a set of relations, most conveniently expressed in frequency space

$$
\begin{aligned}
G_{11} &= \mathrm{Re}G^R + i\coth\left(\frac{\beta\omega}{2}\right)\mathrm{Im}G^R \\
G_{12} &= -2in(\omega)\mathrm{Im}G^R \\
G_{21} &= -2in(\omega)e^{-\beta\omega}\mathrm{Im}G^R \\
G_{22} &= -\mathrm{Re}G^R + i\coth\left(\frac{\beta\omega}{2}\right)\mathrm{Im}G^R
\end{aligned}
\tag{5.42}
$$

with the familiar Bose-Einstein distribution $n(\omega) = (e^{-\beta\omega} - 1)^{-1}$. We recognise exactly the set of Lorentzian correlation functions of Lecture 4, with the choice $\sigma = \beta$. We conclude that the correlation functions obtained from the maximal Kruskal extension of the AdS black hole apparently automatically satisfy the KMS condition together with the FDT. In fact this is consistent with our earlier statement that the black hole represents the thermal equilibrium state, and that the KMS condition is an expression of equilibrium. From a technical point of view this can be traced back to the requirement of having a regular analytic continuation from one side of the black hole to the other. A helpful fact to keep in mind is in fact that every time an equilibrium horizon appears in a holographic problem we have an FDT at the temperature of the horizon.

2. We now make a comment about the physical meaning of the FDT. Recall that the absorptive part $\chi''(\omega)$ of the Green function[4] is given by the anti-Hermitian part

$$\chi''(t,t') := -\frac{i}{2}\left[G^R(t,t') - G^A(t,t')\right] = \mathrm{Im}G^R(t,t')\,.$$

Using Eq. (5.39) and the KMS condition, this can be written as

$$\chi''(t,t') := -\frac{i}{2}\left[G^>(t,t') - G^>(t,t'-i\beta)\right]\,. \tag{5.43}$$

[4]For those unfamiliar with the relationship of the imaginary (or anti-Hermitian) part of the Green function and dissipation, I recommend consulting any standard source, such as Negele & Orland's book, or David Tong's lecture notes on kinetic theory.

After a Fourier transform, we arrive at the expression

$$S(\omega) = -2\,(1 + n(\omega))\,\chi''(\omega)\,, \tag{5.44}$$

where $S(\omega)$ is the Fourier transform of $G^{>}(t,t')$. Had we used fermionic operators in our two-point function, thanks to the additional minus signs, the Fermi-Dirac distribution would have appeared instead. The relation thus tells us that dissipation, as governed by $\chi''(\omega)$ is directly tied to the reactive ('fluctuating') part of the two-point function $S(\omega)$. A famous physical example is furnished by Brownian motion, where the corresponding FDT is called the Einstein relation. It connects Brownian motion (the fluctuating part) to the drag (the dissipative part).

5.4 Path Integral Perspective

We end this chapter by describing the relationship between quantum states and Euclidean path integrals. This will shed more light on the various two-time formalisms we have encountered above, and hopefully illustrate some of the more elaborate constructions encountered in the literature. The main idea is that the role of a Euclidean path integral is to define a state. We can understand this by the procedure of 'cutting open' such path integrals. Consider the transition amplitude from initial state $|\phi_i\rangle$ to final state $|\phi_f\rangle$ evolved along Euclidean time β. This is written as the path integral

$$\langle\phi_f|e^{-\beta H}|\phi_i\rangle = \int \mathcal{D}\phi e^{-S_E[\phi]} \tag{5.45}$$

with boundary conditions $\phi(\tau = 0) = \phi_i$ and $\phi(\tau = \beta) = \phi_f$. One can also view the transition amplitude as defining the wave-function in the Schrödinger picture

$$\Phi[\phi_f] = \langle\phi_f|\phi(\tau)\rangle = \langle\phi_f|e^{-\tau H}|\phi_i\rangle\,. \tag{5.46}$$

That is to say that we think of $|\Phi\rangle = e^{-\tau H}|\phi_i\rangle$ as the ket in the Schrödinger picture at Euclidean time τ. The boundary conditions on the path integral at the initial and final times determine the initial state and what overlap we take with at the

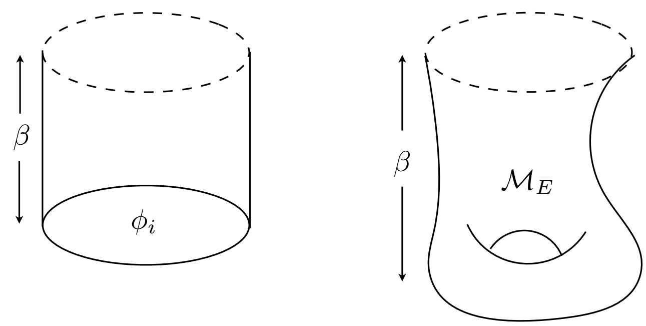


Figure 5.1: The Euclidean path integral with initial condition $\phi(\tau = 0) = \phi_i$ and open boundary conditions at $\phi(\tau = \beta)$ defines a state in the Schrödinger picture. In fact integrating over any Euclidean manifold $\mathcal{M}_E$ with a certain Hamiltonian H defines such a state and we can be as creative as we want in preparing many different interesting states. The right panel shows an example of a genus one surface with no boundary.

final time. From this we deduce the idea of 'cutting open' the path integral: the state at time β is given by a path integral of the form (5.45) where the boundary condition at the final time $\tau = \beta$ is left open (see Figure 5.1), i.e. undetermined.

Once we have prepared such a state, we may do with it whatever we wish. In particular we may use it as an initial state for Lorentzian evolution. The reader is probably already anticipating where we are going with this. Consider then an expectation value of the form

$$\langle \psi(t) \rangle = \langle \phi(t) | \psi | \phi(t) \rangle \tag{5.47}$$

in some state $|\phi(t)\rangle = U(t, t_0)|\phi\rangle$ for a given initial state $|\phi\rangle$. Equivalently we may think of transferring the time dependence to the operator, $\psi(t) = U(t, t_0)\psi U(t_0, t)$. The propagation kernel $U(t, t_0)$ is as usual given by a Lorentzian path integral. Thus the expectation value above is calculated by a path integral over a manifold that has a Euclidean section in order to prepare the state $|\phi\rangle$, a Lorentzian section representing the evolution $U(t_0, t)$, a second Lorentzian section, looping back over the first, representing $U(t, t_0)$ and finally a Euclidean one computing the overlap with $\langle \phi|$. This is illustrated in Figure 5.2.

The different kind of contours we discussed in the operator formalism above are all special cases of this general construction. For example, one may be interested in computing expectation values in the ground state. This is obtained by the

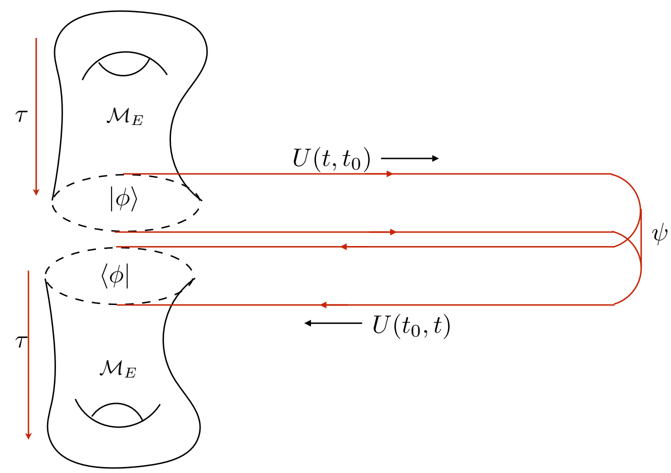


Figure 5.2: A path integral over a manifold obtained from gluing Euclidean sections to Lorenztian sections computes time-dependent expectation values in non-trivial states. The state we are interested in, $|\phi\rangle$, is prepared on the Euclidean section, while the time evolution is achieved by the Lorentzian section. Special cases are the ground state, $|\Omega\rangle$, which we can define by the projection $\beta \to \infty$, or a thermal state, in which case the above construction exactly corresponds to the Kadanoff-Baym contour introduced earlier.

projection

$$|\Omega\rangle \sim \lim_{\beta\to\infty} \sum_n e^{-\beta H}|\phi\rangle \tag{5.48}$$

i.e. by extending the Euclidean section to infinite time. On the other hand, an equilibrium state in the Gibbs ensemble is obtained by evolving along the Euclidean part of the contour of length β with the Hamiltonian $H^M = H - \mu N$, where the superscript 'M' is for Matsubara, and N is the the number operator. In fact we are free to split up the Euclidean part of the evolution in any way we want, so long as the difference in Euclidean time between initial and final time is equal to the inverse temperature β. Performing all of the Euclidean evolution along a single path of length β gives exactly the same construction as the Kadanoff-Baym procedure, while breaking the evolution up into two pieces gives the different $\sigma \in (0, \beta]$ choices we encountered in Chapter 4.

Let us conclude by making a further comment on what has been discussed in Chapter 4. In the context of holography, the operations we described above have a corresponding meaning in the bulk. By this we mean that there is a Euclidean bulk manifold that gets glued to a Lorentzian one and so forth, all individually defining asymptotically AdS geometries. In fact, this construction, at least for the case of real-time thermal averages, predates holography itself and can be traced back to the classic works on black hole physics (see, e.g. [6]). In this context the question arose what state we are computing operator expectation values in, for example when we talk about the fact that a black hole somehow defines a thermal state. One construction, the Hartle-Hawking state in fact corresponds to gluing together a Euclidean version of the black hole to a Lorentzian one about the point of time symmetry. We show the resulting Penrose diagram in Figure 5.3.

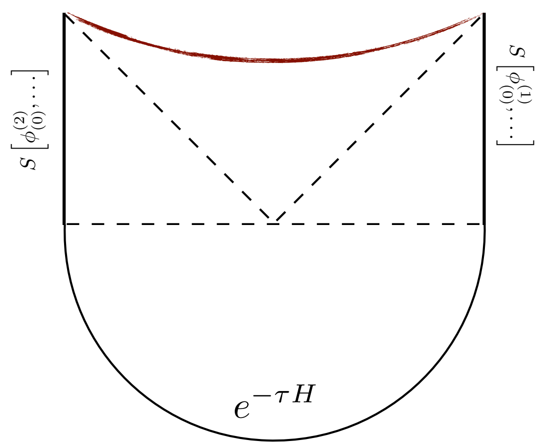


Figure 5.3: The Hartle-Hawking construction of the wavefunction for Schwarzschild AdS.

Bibliography

[1] J. Maciejko, *"An Introduction to Nonequilibrium Many-Body Theory,"* www.physics.arizona.edu/ stafford/Courses/560A/nonequilibrium.pdf

[2] G. Stefanucci and R. Van Leeuwn, *"Nonequilibrium Many-Body Theory of Quantum Systems: A Modern Introduction,"* Cambridge University Press (2013)

[3] J. W. Negele and H. Orland, *"Quantum many-particle systems"*, Addison-Wesley New York (1988)

[4] A. Kamenev, *"Field theory of non-equilibrium systems"*, Cambridge University Press (2011)

[5] G. D. Mahan *"Many-particle physics"*, Springer Science & Business Media (2013)

[6] J. B. Hartle and S. W. Hawking, "*Wave function of the Universe,*" Phys. Rev. D (1983) 12 doi:10.1103/PhysRevD.28.2960,

[7] W. G. Unruh, "*Notes on black-hole evaporation,*" Physical Review D (1976) 4

Chapter 6

Holographic Renormalization

6.1 Holographic Regularization

In the previous chapter we have seen in detail how Lorentzian correlation functions are defined in field theory and how the general framework is represented holographically. We will make use of this Lorentzian framework in the next chapter when we determine transport properties of theories with holographic duals. Before we turn our attention to these applications we shall spend some time to formalize the framework of holographic renormalization[1] that we alluded to briefly further above. For simplicity we will work in Euclidean signature, but the formalism carries over to the Lorenztian signatures, mutatis mutandis (see previous chapter). Our starting point is the relationship between the on-shell gravitational action and the field-theory generating functional of connected correlators,

$$W_{\mathrm{QFT}}[\phi_{(0)}] = -S^{E}_{\text{on-shell}}[\Phi \to \phi_{(0)}]\,. \tag{6.1}$$

We have already seen that we encounter UV divergences in evaluating this expression. By cutting off the integrals in the UV, that is near the AdS boundary, in general, these look like

[1] Our treatment here is based closely on that of [1]. These lecture notes contain more related material and references to the original literature with two classic references being [2] and [3].

$$S_{\text{reg}}[f_{(0)};t] = \int_\epsilon d^d x \sqrt{g_{(0)}} \left[\epsilon^{-\nu} a_{(0)} + \epsilon^{-(\nu-1)} a_{(2)} + \cdots - \log(\epsilon) a_{(2\nu)} + \text{finite}\right] , \quad (6.2)$$

where 'reg' stands for regularized, as we should essentially think of the above expression as a regulated quantity in the field theory sense. We already saw how removing the regulator results in divergent contributions to the free energy, and we also introduced an ad-hoc way to deal with this by noting that physically we are only interested in free-energy differences. However, the divergences also affect $n-$point functions, where it is not obvious that such a background subtraction scheme is particularly well motivated. It will pay to put things on a more solid footing. We will study:

1. How to define a regularized on-shell action $S_{\text{reg}}[\cdots;\epsilon]$
2. How to add counterterms $S_{\text{CT}}[\cdots;\epsilon] = -\text{div}(S_{\text{reg}}[\cdots;\epsilon])$ to the regulated on-shell action
3. How to extract renormalized $n-$point functions from the above which are cutoff independent
4. How to understand RG transformation of $n-$point functions holographically

In order to understand the first item we need to define what we mean by a spacetime that asymptotically looks like anti-de Sitter space. Such a spacetime is also often called asymptotically locally AdS. By extracting the universal behavior of such spaces near the boundary we will be able to define a UV regulator, as required. In order to do so we will introduce some notions from differential geometry of asymptotically anti-de Sitter spaces.

6.1.1 Asymptotic AdS and Fefferman-Graham expansion

The manifold $\mathcal{M}$ =AdS$_{d+1}$ is maximally symmetric, which means that, in a local chart, we can write the Riemann tensor in terms of the metric as

$$R_{abcd} = \frac{1}{\ell^2}\left(G_{ac}G_{bd} - G_{ad}G_{ba}\right). \quad (6.3)$$

It is useful to rewrite the write the global metric (see Eq. (2.10)) as

$$ds^2 = \frac{\ell^2}{\cos^2\theta}\left(-dt^2 + d\theta^2 + \sin^2\theta d\Omega^2_{d-1}\right), \tag{6.4}$$

with $\tan\theta = \sinh\rho$, where $0 \leq \theta < \frac{\pi}{2}$ and ρ is the 'normal' radius of global AdS_{d+1}. We note that (6.4) has a second-order pole at $\theta = \frac{\pi}{2}$, which corresponds in the present coordinate system to UV boundary. However, this observation allows us to strip-off the boundary metric, by compensating this pole with a conformal factor

$$g_{(0)} := r^2 G\big|_{\theta=\pi/2}. \tag{6.5}$$

Strictly speaking, however, this does not define an induced metric on $\partial\mathcal{M}$, but instead a conformal class of boundary metrics, corresponding to the class of functions $r(x^\mu)$ which have a simple zero at the UV boundary. We define

$$r(x) = \begin{cases} r(x) > 0 \text{ in interior} \\ r \text{ has simple zero at } \pi/2 \end{cases}$$

One example would be $r = \cos\theta$. However, if r satisfies the requisite properties to be a defining function, then so does re^ω where ω has no zeros or poles at $\pi/2$. Therefore,

$$[g_{(0)}] = \ell^2\eta_{\mu\nu}dx^\mu dx^\nu \cong \ell^2 e^\omega \eta_{\mu\nu}dx^\mu dx^\nu \tag{6.6}$$

are equally valid boundary metrics, where one is related to the other via a conformal rescaling[2]. This is the precise reason why AdS_{d+1} admits a conformal class of boundary metrics, rather than simply *a particular* boundary metric. Everything that was said up to now referred to the case of anti-de Sitter space itself. We shall now extend the discussion to spaces which merely approach the geometry of AdS when we look at the space near the boundary.

We will take the behavior established above as the model of our definition of an aAdS ('conformally compact' manifold or 'asymptotically locally AdS' manifold).

[2]Whether or not the boundary field theory is invariant under this conformal scaling is an interesting question. If it is not, the theory is said to have a conformal anomaly. See [1] for more details.

Thus, let us define the conformal compactification of a manifold X as the process of equipping the asymptotic boundary with a conformal class of metrics, defining in some sense the closure $\overline{X}$ of the 'interior' X. Thus we define the operation of conformal compactification as

$$\underset{\text{interior}}{X} \longrightarrow \underset{\text{conf. compact.}}{\overline{X}} \tag{6.7}$$

such that the conformal metric $g = r^2 G$ extends smoothly to $\bar{X} \cup \partial X$. By smoothly we mean, of course, that the double pole in the 'bare' metric G is compensated by the double zero in r^2. If we then demand that the metric satisfy the Einstein equations,

$$R_{ab} - \frac{1}{2} R g_{ab} = \Lambda G_{ab} \,, \tag{6.8}$$

we find that the Riemann tensor behaves asymptotically near the UV boundary as

$$R_{abcd} = \frac{1}{\ell^2} \left(G_{ac} G_{bd} - G_{ad} G_{bc} \right) + O(r^{-3}). \tag{6.9}$$

We can choose the function r as the radial coordinate, writing the metric in a near-boundary expansion as

$$ds^2 = \frac{\ell^2}{r^2} \left(dr^2 + g_{\mu\nu}(x, r) dx^\mu dx^\nu \right) \,. \tag{6.10}$$

For the remainder of this chapter it will be convenient to adopt units with $\ell = 1$. The function $g_{\mu\nu}(x, r)$ itself has an expansion of the form

$$g_{\mu\nu}(x, r) = g_{(0)\mu\nu} + r g_{(1)\mu\nu} + r^2 g_{(2)\mu\nu} + \cdots \,, \tag{6.11}$$

where $g_{(A)\mu\nu}$ are determined order by order from (6.8). In the literature often a coordinate $\rho = r^2$ is used, in terms of which we have

$$ds^2 = \frac{d\rho^2}{4\rho^2} + \frac{1}{\rho} g_{\mu\nu}(x, \rho) dx^\mu dx^\nu \tag{6.12}$$

with

$$g(x, \rho) = g_{(0)} + \cdots + \rho^{d/2} g_{(d)} + h_{(d)} \rho^{d/2} \log \rho \,. \tag{6.13}$$

This is known as the Fefferman-Graham expansion after the two mathematicians who first developed it.

Comments

1. We have thus far only considered the vacuum Einstein equations (with cosmological constant), but one can add matter to this process so long as the corresponding matter stress tensor T_{ab} contributes at leading or sub-leading order in the ρ expansion. We will soon see that this corresponds to matter fields which are dual to marginal or relevant operators, respectively, in the boundary field theory. If we added an irrelevant operator, the contribution of the corresponding matter field to the stress energy tensor would be dominant as compared to the cosmological constant. This would render the whole procedure inconsistent and we would have to replace the asymptotically AdS type behavior with a different asymptotic geometry. This is, of course, consistent with expectations from field theory, where irrelevant operators destroy the UV behavior.

2. The point of this differential geometric exercise was to establish a universal notion of the UV behavior of asymptotically AdS geometries. In particular all such manifolds look like AdS near the boundary with small deviations characterized by the higher coefficients $g_{(A)\mu\nu}$. In field theory terms this is the universal behavior of UV fixed points and small deformations about them (marginal, relevant, irrelevant).

3. The present construction also gives us a well-defined notion of UV regulator: Instead of integrating expressions all the way out to the boundary, let us cut off integrals at a radius $\rho = \epsilon$. We should think of this as part of an RG scheme, and we have shown in detail how this part of our scheme is defined for aAdS spaces. This closely parallels the cut off procedure in ordinary RG, such as a momentum cut off or Pauli-Villars or dimensional regularization. Of course the rest of the scheme still needs to be specified.

Let us now specify in the abstract the renormalization scheme most commonly used in the holographic literature. We will then illustrate the procedure with a concrete example, namely a scalar operator dual to a real scalar field in AdS.

6.2 Holographic Renormalization: Step by Step

Since the generating functional involves the gravity partition function with sources, we need to understand the Dirichlet problem in AdS. The Dirichlet data, i.e. the asymptotic behavior of the fields at the boundary, should be thought of as the sources in the field theory. We have already set up the appropriate formalism that tells us how to thing about this: In order to give Dirichlet boundary conditions for the metric g, we should specify $g_{(0)\mu\nu}(x^\mu)$, as an arbitrary function of the boundary coordinates.

The bulk field content, mirroring the operator content of the dual field theory, contains modes with near boundary expansions of the form

$$\mathcal{F}(x,\rho) = \rho^m \left(f_{(0)}(x) + f_{(2)}(x)\rho + \cdots + \rho^n [f_{(2n)}(x) + \log\rho \tilde{f}_{(2n)}(x) + \cdots] \right) . \tag{6.14}$$

These operators could be bosonic or fermionic and carry various interesting representations of the Lorentz group (scalar, spinor, vector, tensor, ...). We have already encountered an example in Eq. (6.11) above for the metric. The exponents m and n are determined via an asymptotic analysis of the bulk equations of motion. Let us proceed with the generic form and explain the meaning of the individual terms.

Firstly, the leading behavior, $f_{(0)}$, is the Dirichlet data piece and acts as a 'source' for the operator $\mathcal{O}_\mathcal{F}$ in the dual field theory. For the case of the metric in Eq. (6.11) the data $g_{(0)\mu\nu}$ acts as a source for the energy momentum tensor T_{ij} in the boundary field theory.

The sub-leading pieces $f_{(w)}, \ldots, f_{(2n-2)}$ are determined by the bulk equations of motion. These also determine the coefficient $\tilde{f}_{2n}$, which is related to the conformal anomaly. A theory with non-vanishing $\tilde{f}_{2n}$ is not invariant under a change of representative of the conformal class of the boundary metric. In other words, if we perform a conformal transformation of the boundary metric, correlation functions or free energies pick up an anomalous piece. The information contained in these expansion coefficients is enough to determine the divergent coefficients $a_{(A)}$ in (6.2) in terms of $f_{(0)}$, that is we have $a_{(A)}[f_{(0)}]$. This allows us to write down the regularized action, isolating the pieces which diverge when the cutoff is removed.

We then proceed to the next step in our renormalization procedure, the addition of counterterms. These are designed so as to cancel exactly the divergent pieces in the regularized action. One therefore has the subtracted action

$$S_{\text{sub}} = S_{\text{reg}}[f_{(0)}, \epsilon] + S_{\text{ct}}[f_{(0)}, \epsilon], \tag{6.15}$$

where

$$S_{\text{ct}}[f_{(0)}, \epsilon] = -\text{div}\left(S_{\text{reg}}[f_{(0)}, \epsilon]\right) . \tag{6.16}$$

It may be a useful analogy to compare this to the 'minimal subtraction' scheme in dimensional regularization of quantum field theories. In fact, I will refer to this procedure as 'holographic minimal subtraction'. The resulting object is finite when the cutoff is removed and defines the generating functional of renormalized correlation functions

$$S_{\text{ren}} = \lim_{\epsilon \to 0} S_{\text{sub.}}[f_{(0)}, \epsilon] = -W^{\text{ren.}}_{\text{QFT}}[f_{(0)}] . \tag{6.17}$$

Differentiating with respect to sources generates as usual these correlation functions. For example

$$\langle \mathcal{O}_{\mathcal{F}} \rangle = \frac{1}{\sqrt{g_{(0)}}} \frac{\delta S_{\text{ren}}}{\delta f_{(0)}} \sim f_{(2n)}. \tag{6.18}$$

The exact coefficient depends on normalization choices and such details and we will determine it for the simplest case of a scalar operator in the next section.

Comment

The coefficient $f_{(2n)}$, which is dual to the expectation value of the operator $\mathcal{O}_{\mathcal{F}}$ is not determined by near boundary analysis. In order to determine it we need full solution and in particular an infra-red boundary condition which together with the Dirichlet condition in the UV uniquely determines the solution. Let us now see how this somewhat abstract discussion plays out in the simplest example, namely a spinless bosonic operator $\mathcal{O}_\Phi$ dual to a bulk scalar field $\Phi(\rho, x)$.

6.2.1 Example: Scalar Operator

A spinless bosonic operator is dual to a real scalar field in the bulk. We have the dual relationships

$$\text{scalar field } \Phi \leftrightsquigarrow \text{spinless operator } \mathcal{O}_\Phi$$

$$\text{mass } m \leftrightsquigarrow \text{conformal dim. } \Delta_\Phi$$

The bulk scalar field obeys equations of motion following from the action[3]

$$S = \frac{1}{2} \int d^{d+1}x \sqrt{G} (G^{ab} \partial_a \Phi \partial_b \Phi + m^2 \Phi^2) \,. \tag{6.19}$$

Explicitly, they take the form of a wave equation, viz.

$$(-\Box_G + m^2)\Phi = -\frac{1}{\sqrt{G}} \partial_a (\sqrt{G} G^{ab} \partial_b \Phi) + m^2 \Phi = 0 \,. \tag{6.20}$$

In fact we should really solve the coupled system $S_{\text{grav}} + S[\Phi]$, i.e. the Einstein equations coupled to a scalar field. For simplicity here we use the so-called probe approximation, that is we neglect the backreaction of the scalar field on the geometry. In fact, on the level of the two-point function, this gives us the full result. So let us find a solution that behaves asymptotically as $\Phi = \rho^a \phi(x, \rho)$. With foresight, let us parametrize this exponent as $a = \frac{1}{2}(d - \Delta)$. The field $\phi(x, \rho)$ has a regular expansion in power of ρ,

$$\phi(x, \rho) = \phi_{(0)} + \rho \phi_{(2)} + \rho^2 \phi_{(4)} + \cdots \,. \tag{6.21}$$

As usual the exponent a, or equivalently the constant Δ are determined by the indicial equation, which in the present case reads

$$m^2 - \Delta(\Delta - d) = 0 \tag{6.22}$$

[3]Recall that we are using Euclidean signature in this chapter. As we have seen several times now, the main subtlety lies in the infra-red boundary condition, and we avoid this by sticking to the Euclidean case. The near-boundary analysis carries over unchanged to Lorentzian signature and so in particular the divergences are treated in exactly the same way.

In fact, we chose the notation in terms of Δ since this equation determines conformal dimension of the operator $\mathcal{O}_\Phi$ in terms of the bulk mass m. We shall return to this interpretation below. Having stripped off this asymptotic behavior, the remaining equation for $\phi(x,\rho)$ is

$$0 = \underbrace{\delta^{ij}\partial_i\partial_j\phi}_{\Box_0\phi} + 2(d - 2\Delta + 2)\partial_\rho\phi + 4\,\rho\,\partial_\rho^2\phi\,. \tag{6.23}$$

This equation can now be solved straightforwardly, order by order in the ρ expansion. At leading order we find that the coefficient $\phi_{(2)}$ is fully determined in terms of the source,

$$\phi_{(2)} = \frac{1}{2(2\Delta - d - 2)}\Box_0\phi_{(0)}\,. \tag{6.24}$$

Carrying out this process to higher order results in the expressions

$$\begin{aligned}
\phi_{(2)} &= \frac{1}{2(2\Delta - d - 2)}\Box_0\phi_{(0)} \\
\phi_{(4)} &= \frac{1}{4(2\Delta - d - 4)}\Box_0\phi_{(2)} \\
&\vdots \\
\phi_{(2n)} &= \frac{1}{2n(2\Delta - d - 2n)}\Box_0\phi_{(2n-2)}
\end{aligned} \tag{6.25}$$

where, crucially, all these coefficients are determined algebraically in terms of the source. In other words, once the Dirichlet data is specified, all the above coefficients are uniquely determined. Attempting to push the expansion further, we need to distinguish between two cases.

I) So long as $2\Delta - d$ is not an integer, the second solution with leading power $\rho^{\Delta/2}\phi_{2\Delta-d}$ is not determined in terms of the source $\phi_{(0)}$. Another way of saying the same thing is that the two asymptotic solution branches are linearly independent. As we shall see, the coefficient $\phi_{2\Delta-d}$ plays the role of he expectation value of the dual operator. As before this should not be determined by the asymptotic analysis alone, but instead we need infra-red input.

II) If, however, $2\Delta - d$ is integer, the solutions with power $\rho^{\Delta/2}$ and $\rho^{\frac{1}{2}(d-\Delta)}$ are linearly dependent. Thus, in order to have two linearly independent solutions – as is required for a second-order equation – it is necessary to introduce a logarithmic term at order $O(\rho^{\Delta/2})$:

$$\rho^{\Delta/2}\left(\phi_{(2\Delta-d)}+\psi_{(2\Delta-d)}\log\rho+\cdots\right), \tag{6.26}$$

with

$$\psi_{(2\Delta-d)}=-\frac{1}{2^{2k}\Gamma(k)\Gamma(k+1)}(\Box_0)^k\phi_{(0)},\,. \tag{6.27}$$

Again the expectation value $\phi_{(2\Delta-d)}$ is not determined by $\phi_{(0)}$. The logarithmic coefficient $\psi_{(2\Delta-d)}$ contributes to the conformal anomaly $\mathcal{A}$ of the dual field theory.

We now have determined the solutions in an expansion near the UV boundary, leaving undetermined one of the two linearly independent solution branches. This structure contains all the necessary information in order to determine the regularized action, and in particular the divergent part. We evaluate the on-shell action in terms of our asymptotic series, arriving at the expression

$$\begin{aligned}S_{\text{reg}} &= \frac{1}{2}\int_{\rho\geq\epsilon} d^{d+1}x\sqrt{G}\left((\partial\Phi)^2+m^2\Phi^2\right)\\ &= \int_{\rho=\epsilon} d^dx\left(\epsilon^{-\Delta+\frac{d}{2}}a_{(0)}+\epsilon^{(-\Delta+\frac{d}{2}+1)}a_{(2)}+\cdots\right),\end{aligned}$$

where the coefficients of the divergent pieces are determined by the asymptotic expansion as local functions of the source

$$a_{(0)}=-\frac{1}{2}(d-\Delta)\phi_{(0)}^2;\qquad a_{(2)}=-\frac{d-\Delta+1}{2(2\Delta-d-2)}\phi_{(0)}\Box_0\phi_{(0)}.$$

Next on our to-do list is to find the corresponding local counterterms that cancel these divergences. In order to achieve this, it is easiest to just invert the asymptotic expansion to a given order. We first state the resulting counterterms and carry out the calculation later. The statement is then that all divergences present in the regularized action are cancelled by the counterterms

$$S_{\mathrm{ct}} = \int \sqrt{\gamma} \left(\frac{d-\Delta}{2} \Phi^2 + \frac{1}{2(2\Delta - d - 2)} \Phi \Box_\gamma \Phi \right) . \tag{6.28}$$

Then we define the renormalized

$$S_{\mathrm{ren}} = \lim_{\epsilon \to 0} \left(S_{\mathrm{sub}} - S_{\mathrm{ct}} \right) \tag{6.29}$$

which has, by construction, a finite limit as the cutoff is removed. A calculation shows that

$$\frac{\delta S_{\mathrm{ren}}}{\delta \phi_{(0)}} := \langle O_\Phi \rangle = -(2\Delta - d)\phi_{(2\Delta - d)} + C(\phi_{(0)}) \,, \tag{6.30}$$

where the last term is the scheme-dependent part, which can be removed with finite counter-terms. The expectation value, as presaged several times above is determined by the coefficient $\phi_{(2\Delta-d)}$, which itself is *not* determined by the Dirichlet data $\phi_{(0)}$ at the boundary. It will be determined from matching the UV behavior to the appropriate IR solution in order to construct the full bulk field. We now go back to the details of the counterterm action, supplying the calculational details we skipped above. We want to invert

$$\Phi = \rho^{\frac{1}{2}(d-\Delta)} \left(\phi_{(0)} + \frac{\rho}{2(2\Delta - d - 2)} \Box_0 \phi_0 + \cdots \right) \tag{6.31}$$

at $\rho = \epsilon$. This is straightforward and we find

$$\Phi_\epsilon = \epsilon^{\frac{1}{2}(d-\Delta)} \left(\phi_{(0)} + \frac{\epsilon}{2(2\Delta - d - 2)} \Box_0 \phi_0 + \cdots \right) . \tag{6.32}$$

Therefore

$$\begin{aligned}\phi_{(0)} &= \epsilon^{\frac{\Delta}{2}-\frac{d}{2}}\Phi_\epsilon - \frac{\epsilon}{2(2\Delta-d-2)}\Box_0\phi_{(0)} + \cdots \\ &= \left(\Phi_\epsilon - \frac{1}{2(2\Delta-d-2)}\underbrace{\epsilon\Box_0}_{\Box_\gamma}\Phi_\epsilon + \cdots\right),\end{aligned}$$

where $\epsilon\,\Box_{(0)} = \epsilon\,\delta^{ij}\partial_i\partial_j$ is the Laplacian[4] with respect to the metric $\gamma_{ij} = \frac{1}{\epsilon}\delta_{ij}$ induced on the cutoff surface. Similarly for $\phi_{(2)}$ one has

$$\begin{aligned}\phi_{(2)} &= \frac{1}{2(2\Delta-d-2)}\Box_0\phi_{(0)} \\ &= \frac{\epsilon^{\frac{\Delta}{2}} - \frac{d}{2}}{2(2\Delta-d-2)}\Box_0\Phi_\epsilon + O(\Box^2\Phi_\epsilon) \\ &= \frac{\epsilon^{\frac{\Delta}{2}} - \frac{d}{2} - 1}{2(2\Delta-d-2)}\Box_\gamma\Phi_\epsilon + \cdots .\end{aligned}$$

Substituting these expressions into the regularized action we find that the divergent pieces are exactly (minus) the counterterms shown in Eq. (6.28) above. This proves that the subtracted action indeed is finite upon removal of the cutoff, as required.

6.2.2 Callan-Symanzik Equation

The renormalized action (6.29) defines for us a generating functional for renormalized correlation functions. As usual it is illuminating to study the behavior of the renormalized quantities under a scale transformation. Such a scale transformation acts holographically as

$$\rho \longrightarrow \mu^2\rho : \qquad \text{energy scale} \tag{6.33}$$

$$x^i \longrightarrow \mu x^i : \qquad \text{scale transformation}\,, \tag{6.34}$$

[4]If the signature is Lorentzian it would be the d'Alambertian

where the power of μ has been fixed by dimensional analysis following our UV analysis. The idea is to find the behavior of the various coefficients in the UV expansion – keeping in mind their interpretation as sources and expectation values in the dual field theory – and then finding the differential equations satisfied by these objects as a function of scale. Such equations are referred to in the literature as Callan-Symanzik equations, and we find here their holographic incarnations. Let us proceed by considering the simplest case, namely that of a spinless bosonic operator, dual to a scalar field in the bulk. By definition, the scalar field $\Phi(x,\rho)$ is invariant under these transformation, i.e.

$$\Phi'(x',\rho') = \Phi(x,\rho)\,. \tag{6.35}$$

By studying the action of the transformation on each term in the expansion (6.21), and imposing the transformation law (6.35), one finds the induced transformations of the coefficient functions. We display the two most interesting ones. The source transforms as

$$\phi'_{(0)}(x') = \mu^{d-\Delta}\phi_{(0)}(x'\mu) \tag{6.36}$$

while the expectation value transforms as

$$\phi'_{(2\Delta-d)}(x') = \mu^{\Delta}(\phi_{(2\Delta-d)}(x'\mu) + \underset{\substack{\text{if log term}\\ \text{present}}}{\log}\mu^2\psi_{(2\Delta-d)}(x'))\,, \tag{6.37}$$

where, as indicated, the inhomogeneous part of the transformation only occurs in cases where there is a log term in the Fefferman-Graham expansion. By taking derivatives with respect to the scale we deduce from (6.36)

$$\mu\frac{\partial}{\partial\mu}\phi_{(0)}(x'\mu) = -(d-\Delta)\phi_{(0)}(x'\mu), \tag{6.38}$$

which expresses the fact that $\mathcal{O}_\Phi$ has dimension Δ. In fact, this can be seen even more clearly by finding the RG equation satisfied by the expectation value. Taking derivatives with respect to scale we have

$$\langle\mathcal{O}(x')\rangle = \mu^{\Delta}\left[\langle\mathcal{O}(x'\mu) - (2\Delta-d)\log\mu^2\psi_{(2\Delta-d)}(x'\mu)\right]. \tag{6.39}$$

That is, the one-point function scales as μ^Δ *up to* possible logarithmic corrections due to *anomaly*. Disregarding this anomalous piece we have precisely the transformation law for the one-point function of an operator of dimension Δ. It should be evident by now that higher n-point functions will satisfy exactly the right RG equations to qualify as n-point functions of operators of dimension Δ, by a procedure analogous to the above.

6.3 Witten Diagrams

In general the computations we have performed are sensitive to the interaction terms that appear in the bulk action representing the dual field theory. So far, for simplicity, we have only really considered the case of a free massive scalar, but as we already commented, for a full treatment we should also take into account its interaction with the gravitational sector. Naturally any (set of) bulk field may also have its self-interactions or interaction vertices with other bulk fields. To convince oneself of this fact, I encourage the reader to briefly consider the case of a non-abelian Yang-Mills field in the bulk which obviously has gluon-gluon vertices in the bulk. We will now outline how bulk interactions are featured in the formalism above. In fact, this is not a conceptual complication, but can lead to a considerable calculational complication. Let us again illustrate what is going on via the simple exmaple of bulk scalar interactions. The algebraic expressions we encounter are liable to become rather unwieldy, so at an appropriate point we will introduce a diagrammatic representation analogous to Feynman diagrams which helps to easier visualize the essential physics.

6.3.1 Bulk Scalar With Self Interaction

For this purpose we generalize our action (6.19) to

$$S = \int d^{d+1}x \left(\sum_{i=1}^{3} \tfrac{1}{2} \left(\partial \Phi_i\right)^2 + \tfrac{1}{2} m^2 \Phi_i^2 + \tfrac{1}{3} \lambda_{ijk} 3 \Phi_i \Phi_j \Phi_k \right) , \tag{6.40}$$

that is we consider a multiplet of three scalar operators $\mathcal{O}_{\Phi_i}$, all with the same conformal dimension Δ which interact via a bulk three-point coupling. The idea is now to solve the equations of motion

$$\left(-\Box + m^2\right)\Phi_i + \lambda_{ijk}\Phi_j\Phi_k = 0 \tag{6.41}$$

perturbatively in $\lambda_{ijk} \sim \mathcal{O}(\lambda)$ with $\lambda \ll 1$. That is we seek

$$\Phi_i = \Phi_i^{(0)} + \Phi_i^{(1)} + \mathcal{O}(\lambda^2) \tag{6.42}$$

where the fields have prescribed boundary conditions

$$\Phi_i \sim \rho^{\frac{d-\Delta}{2}}\phi_i^{(0)} + \cdots . \tag{6.43}$$

We have the equations

$$\begin{aligned}\left(-\Box + m^2\right)\Phi_i^{(0)} &= 0\,,\\ \left(-\Box + m^2\right)\Phi_i^{(1)} &= 2\lambda_{ijk}\Phi_j^{(0)}\Phi_k^{(0)}\,.\end{aligned} \tag{6.44}$$

In order to solve them, we introduce two formal objects. Firstly we introduce the bulk to boundary propagator, $K_\Delta^{ij}(\rho, x^\mu - y^\mu) = \delta^{ij}K_\Delta(\rho, x^\mu - y^\mu)$, satisfying

$$\left(-\Box + m^2\right)K_\Delta(\rho, x^\mu - y^\nu) = \rho^{\frac{d-\Delta}{2}}\delta^d(x^\mu - y^\mu)\,, \tag{6.45}$$

and secondly the bulk to bulk propagator

$$\left(-\Box + m^2\right)G_\Delta(x^a - y^a) = \delta^{d+1}(x^a - y^a)\,. \tag{6.46}$$

The former allows us to write the full bulk solution at order λ^0 as a function of the boundary value

$$\Phi_i^{(0)}(\rho, x^\mu) = \int d^d y K_\Delta\left(\rho, x^\mu - y^\mu\right)\phi_i^{(0)}(y^\mu)\,, \tag{6.47}$$

while the latter allows us to write the full bulk solution at first order in λ as

$$\Phi_i^{(1)}(\rho, x^\mu) = \lambda_{ijk}\int d^{d+1}y d^d u d^d v G_\Delta(x^a - y^a)K_\Delta(z, y^\mu - u^\mu)\phi_j^{(0)}(u^\mu)K_\Delta(z, y^\mu - u^\mu)\phi_k^{(0)}(v^\mu)\,. \tag{6.48}$$

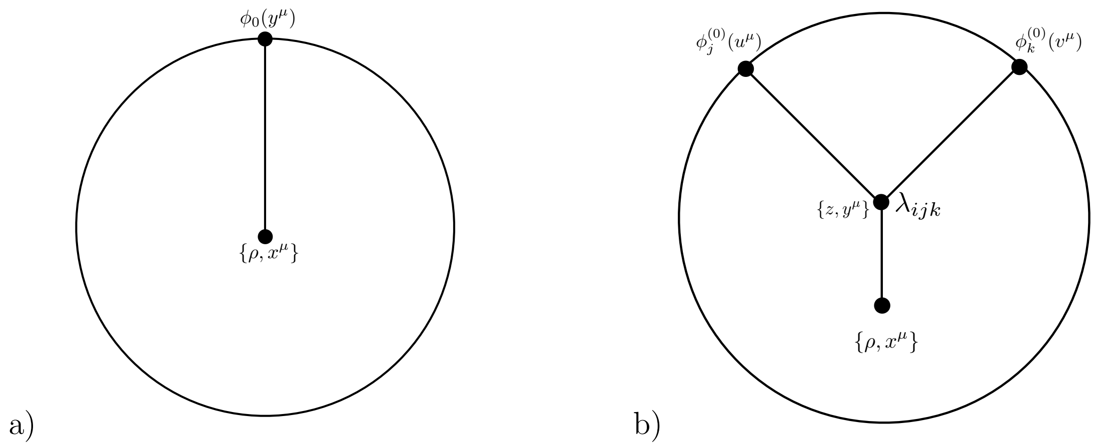


Figure 6.1: Two (incomplete) Witten diagrams showing the two basic contributions ((6.47) in panel a) and ((6.3.1) in panel b) to the bulk solution $\Phi_i(\rho, x^\mu)$. These form the basic building blocks for the so-called Witten diagrams used in the calculation of higher-point functions in holography. Note that all source insertions as well as bulk vertices are integrated over.

The first equation is very familiar from our previous two-point function calculations. There we calculated an explicit representation of the bulk-to boundary propagator in momentum space in terms of the mode functions f_k. Here we do not enter into the specifics and instead work with the abstract object K_Δ itself. The second equation expresses the field at an arbitrary bulk point $x^a = (\rho, x^\mu)$ in terms of an integral over the two sources at the boundary. One is inserted at the boundary point u^μ and then propagated into the bulk point $y^a = (z, y^\mu)$ with $K_\Delta(z, y^\mu - u^\mu)$, the second is inserted at the boundary at the point v^μ and then propagated to the same bulk point $y^a = (z, y^\mu)$ with $K_\Delta(z, y^\mu - v^\mu)$. The final ingredient is a further propagation in the bulk via the bulk interaction λ_{ijk} from the point $y^a = (z, y^\mu)$ to the point $x^a = (\rho, x^\mu)$ using the bulk to bulk propagator $G_\Delta(x^a - y^a)$. It is probably best to think about these equations visually, via so called Witten diagrams, which are a holographic analog of Feynman diagrams in ordinary field theory. The two contributions in Eq. (6.47) and (6.3.1) are shown in Figure 6.1.

To go from here to the correlation functions only requires a bit little more work. The bulk solution (6.47)(6.3.1) has the asymptotic behavior of a field of mass m in anti-de Sitter space, so from our holographic RG analysis we know that the renormalized one-point function is given as the expansion coefficient $\phi_{2\Delta-d}$, up to contact terms, the precise relation being given by Eq. (6.30). This means that we extract the one-point function in the presence of sources simply by reading off the appropriate coefficient in the near-boundary expansion of the solution (6.47), (6.3.1) . But once we have the renormalized one-point function in the presence of sources we can generate all higher point functions by further differentiation with respect to sources. In the present context, for example, the four-point function will be given by

$$\langle \mathcal{O}_i(x_1^\mu)\mathcal{O}_j(x_1^\mu)\mathcal{O}_k(x_1^\mu)\mathcal{O}_l(x_1^\mu)\rangle = \left.\frac{\delta^3 \langle \mathcal{O}_i\rangle_{\rm ren}}{\delta\phi_j^{(0)}\delta\phi_k^{(0)}\delta\phi_l^{(0)}}\right|_{\phi_i^{(0)}=0} . \tag{6.49}$$

Rather than carrying out this tedious computation explicitly, let us display some contributing Witten diagrams in Fig. 6.2. As a final comment, let us remark that, in principle, this procedure can be pushed beyond tree level, where we treat the bulk gravity solution as an effective field theory (EFT). This way we may compute loop corrections to any n-point function, as illustrated in Fig. Corrections that involve graviton vertices are suppressed by appropriate factors of the bulk Newton constant $G_N \sim 1/N^2$. It should by now be apparent that this procedure produces for us any Euclidean n-point function.

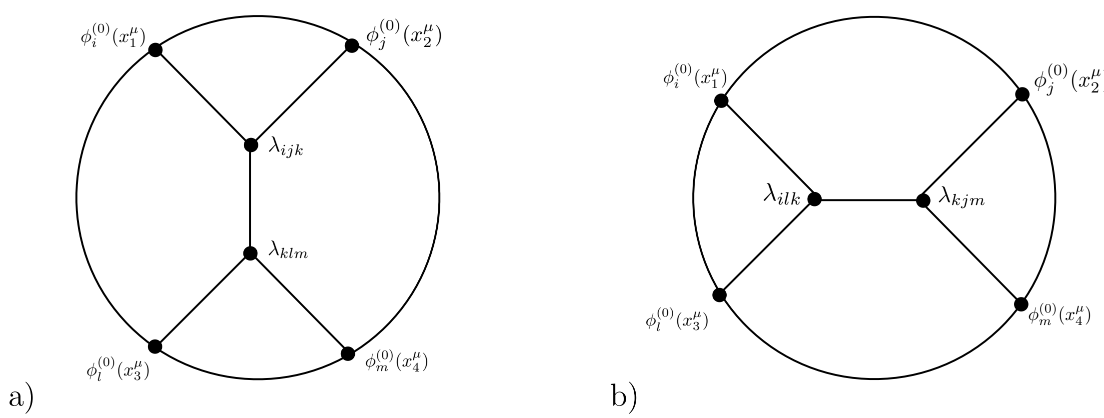


Figure 6.2: Two tree-level Witten diagrams contributing to the boundary four-point function $\langle \mathcal{O}_i\mathcal{O}_j\mathcal{O}_l\mathcal{O}_m\rangle$. In general all bulk interactions must be taken into account, including couplings to the graviton and all other modes of the spectrum, as well as a sum over all exchange channels.

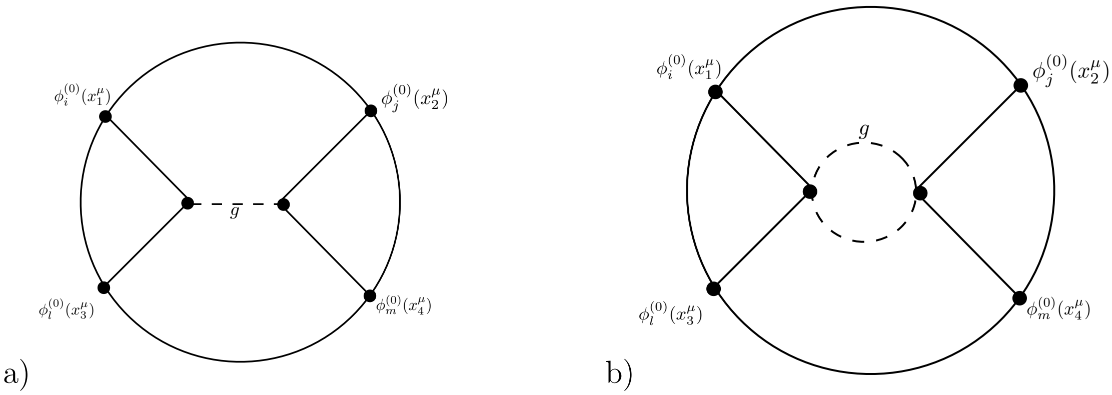


Figure 6.3: Two Witten diagrams contributing to the boundary four-point function $\langle \mathcal{O}_i \mathcal{O}_j \mathcal{O}_l \mathcal{O}_m \rangle$ involving bulk graviton exchange. The first contribution (panel a) is at tree level, the second (panel b) at one loop. In general these contribute to the correlation function at order $\frac{1}{M_{\mathrm{P}}^{2+2L}}$, where L is the number of loops and M_{P} is the Planck mass. For the example of the $\mathcal{N} = 4$ theory we have $M_{\mathrm{P}} \sim N$. Of course all bulk fields may run in loops and they may come with their own small coupling parameter (such as λ above). The loop counting proceeds as in ordinary EFT.

6.4 Lorentzian Formalism

As usual we could construct Lorentzian correlation functions by suitable analytic continuations from the Euclidean ones, but for various reasons discussed previously it is useful to have a natively Lorentzian framework for the computation of correlation functions. As we have seen, this is provided by the various incarnations of the two-time formalism. The holographic realization of the contour idea means that we need to find 'infilling' solutions for the various different contours (e.g. the Schwinger-Keldysh contour or the Kadanoff-Baym contour, etc.). As we saw above and in particular from the path-integral point of view, this involves the construction of manifolds that contain both Euclidean and Lorentzian sections, glued together at common boundaries.

In this case the semi-classical saddle point of the gravity action looks like

$$Z\left[\Phi_i \to \phi_i^{(0)}\right] \sim e^{-iS_1 + iS_2 - S_{\mathrm{E}}}\,,$$

where S_{E} represents the contribution from the Euclidean parts of the solution (and may or may not be present, according to the contour we are using) and

$S_{1,2}$ are the two Lorentzian parts. The crucial point to appreciate is that the asymptotic structure near each aAdS boundary is exactly the same our previous Euclidean analysis, so long as we replace, in each case, the Laplacian $\Box$ with the d'Alambertian $\Box_{dA}$ in Lorentzian signature. Except for this trivial replacement the asymptotic analysis for each region goes through exactly as before. The reason for this is that all asymptotic coefficients needed to determine the counterterms followed *algebraically* in the previous analysis. Of course, there are potential further subtleties arising from the gluing procedure as well as from additional time-like boundaries that arise at the end of the 'vertical' parts of the contours in the in-in formalism, but once the dust settles one can show that these do not lead to further divergences. The upshot is thus that one adds the usual counterterm action for each part of the contour (and infilling solution) that has an aAdS boundary, which defines a finite renormalized generating functional for Lorenztian correlation functions out of equilibrium. We show an explicit example of such a gluing in Fig. 6.4.

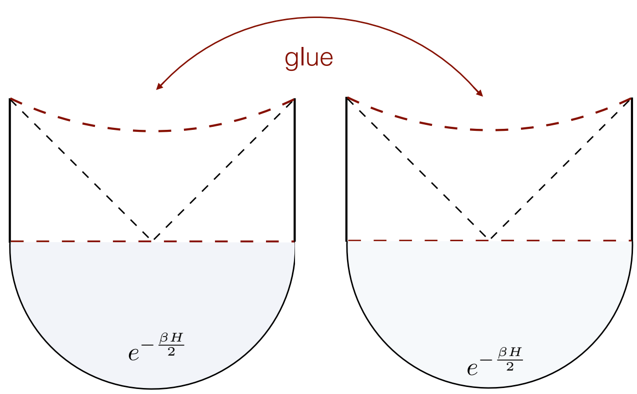


Figure 6.4: An example of a glued bulk manifold. Here we take the AdS black hole (e.g. BTZ to be specific). The two rounded sections each correspond to half the Euclidean black hole (in light gray), each then glued to a Lorentzian section of the black hole at the instant of time symmetry. The manifold is glued together along the dashed (red) lines. The gluing procedure shown here in fact naturally produces the thermal contour with $\sigma = \beta/2$: we have in fact split up the Euclidean evolution into two halves as shown. This is the natural gluing realizing Lorentzian $n-$point functions in the Hartle-Hawking state. It neatly explains why the naive gravity calculation in the Hartle-Hawking state yields he field theory correlations for the contour choice with $\sigma = \beta/2$.

$$W_{\text{ren}}\left[\phi_1^{(0)},\phi_2^{(0)}\right] := \log Z_{\text{ren}}\left[\phi_1^{(0)},\phi_2^{(0)}\right]\,. \tag{6.50}$$

Note that here we only allowed for sources on the Lorentzian parts of the contour. This then leads to well defined renormalized n-point functions

$$iG_{a_1\ldots a_n}\left(x_1,\ldots,x_n\right) = \frac{1}{i^n}\frac{\delta^n W_{\text{ren}}\left[\phi_1^{(0)},\phi_2^{(0)}\right]}{\delta\phi_{a_1}^{(0)}\cdots\delta\phi_{a_n}^{(0)}}\,. \tag{6.51}$$

The solutions in the bulk must now be matched regularly across the various joins in the manifold, reflected in the bulk-to-bulk (6.46) and bulk-to boundary propagators (6.45). In fact these manipulations are standard hailing back to the glory days of black hole thermodynamics (see e.g. Ref. [7] of Chapter 5), and ported in a detailed fashion to the holographic context in [4, 5, 6]. We refer the interested reader there for more details. We have now treated, in some detail the most general framework to calculate arbitrary renormalized n-point functions in holography, generalizing the original recipe for two-point functions of Son and Herzog to arbitrary initial states, including those far from equilibrium. We close with the comment that for most practical application thus far, only low-order correlations (actually one- and two-point functions) near equilibrium (i.e. around a black hole background) are needed. As we reviewed in detail in this case we have a fluctuation-dissipation relation, which allows us to reconstruct the full set of two-point functions from the knowledge of only one Lorentzian asymptotic region. This explains why the majority of calculations in the literature do not involve the more complicated formalism described here. It is hoped that the somewhat complicated treatment in this chapter has served to put this formalism on solid footing, removing the ad-hoc-ness of the usual presentation. It is also hoped that the more adventurous among you, equipped with the full story, embark on an analysis of the many exciting physical results that await to be discovered in the fully non-equilbrium regime where the present framework is indispensable.

Bibliography

[1] K. Skenderis, *"Lecture Notes on Holographic Renormalization,"* Class. Quant. Grav. **19** (2002) 5849 doi:10.1088/0264-9381/19/22/306 [hep-th/0209067].

[2] V. Balasubramanian and P. Kraus, *"A Stress Tensor for Anti-de Sitter Gravity,"* Commun. Math. Phys. **208** (1999) 413 doi:10.1007/s002200050764 [hep-th/9902121].

[3] S. de Haro, S. N. Solodukhin and K. Skenderis, *"Holographic Reconstruction of Space-Time and Renormalization in the AdS / CFT Correspondence,"* Commun. Math. Phys. **217** (2001) 595 doi:10.1007/s002200100381 [hep-th/0002230].

[4] C. P. Herzog and D. T. Son, *"Schwinger-Keldysh Propagators from AdS/CFT Correspondence,"* JHEP **0303** (2003) 046 doi:10.1088/1126-6708/2003/03/046 [hep-th/0212072].

[5] D. T. Son and D. Teaney, *"Thermal Noise and Stochastic Strings in AdS/CFT,"* JHEP **0907** (2009) 021 doi:10.1088/1126-6708/2009/07/021 [arXiv:0901.2338 [hep-th]].

[6] K. Skenderis and B. C. van Rees, *"Real-Time Gauge/Gravity Duality,"* Phys. Rev. Lett. **101** (2008) 081601 doi:10.1103/PhysRevLett.101.081601 [arXiv:0805.0150 [hep-th]].

Chapter 7

Strongly Coupled Transport

7.1 Transport in 2+1

Transport concerns the study of how physical systems – mostly in the context of condensed matter – respond to external forces via currents. Familiar examples include the conductivity which quantifies how charge is moved in the form of an electric current in response to an external electric field. Similarly heat conductivity quantifies the momentum current in response to an applied heat gradient. Finally, viscosity measures momentum transport as a response to an applied shear. It is fair to say that the most famous result of applied to holography to date is the universal shear-viscosity to entropy ratio first calculated by Policastro, Son and Starinets for the $\mathcal{N} = 4$ SYM theory [1]. In this chapter we will bring some of the technology we developed in previous chapters to bear on a variant of this calculation. In particular we will calculate the shear viscosity of a $2+1$ strongly coupled quantum field theory, dual to a $3+1$ dimensional asymptotically AdS geometry. Since the result will be universal for this class of theories, we do not need to specify exactly which boundary theory we have in mind, but it may still be helpful to note that a concrete top down example exists in the form of the ABJM superconformal field theories, dual to a stack of M2 branes. This calculation was first done in [2], before the advent of the ABJM theories.

Let us recall that the (shear) viscosity is given by a Kubo formula :

$$\eta = \lim_{\omega \to 0} \left[\lim_{q \to 0} \frac{1}{\omega} G^R_{xy,xy}(\omega, q) \right] , \tag{7.1}$$

where $G^R_{xy,xy}(\omega, q)$ refers to specific components of the stress tensor two point function,

$$G^R_{\mu\nu,\rho\sigma}(t, \vec{x}) = \Theta(t) \langle [T_{\mu\nu}(t, \vec{x}), T_{\rho\sigma}(0, 0)] \rangle$$

in momentum space. Similarly there is a Kubo formula for conductivity

$$\sigma = \lim_{\omega \to 0} \left(\lim_{q \to 0} \frac{1}{\omega} G_{J_x J_x}(\omega, q) \right) , \tag{7.2}$$

where now we compute the retarded current-current two point function

$$G^R_{J_\mu j_\nu}(t, \vec{x}) = \Theta(t) \langle [J_\mu(t, \vec{x}), J_\nu(0, 0)] \rangle.$$

Note that in general these are complex quantities, but our main interest here is in their imaginary parts. Referring back to the chapter on linear response the reader can verify that this is equivalent to focusing on the *real* part of the conductivity matrix, $\mathrm{Re}\sigma_{ij}$. Furthermore, one can consider these quantities as functions of frequency and momentum, but what one usually calls *the* conductivity is the DC conductivity, that is the limit where momentum goes to zero and then frequency goes to zero of the more general object. Similarly for *the* shear viscosity.

7.1.1 Charge Diffusion & Conductivity

We begin with the calculation of charge transport (7.2). A conserved current in the boundary field theory is dual to a U(1) gauge field in the bulk

$$J_\mu \quad \leftrightsquigarrow \quad \text{gauge field } A_a(x^b)$$

$$G^R_{J_\mu J_\nu} \quad \leftrightsquigarrow \quad \text{ingoing modes of } A_a(x^b) \tag{7.3}$$

with action

$$S_{\text{current}} = -\frac{1}{\ell^2}\int d^4x\sqrt{-g}F_{ab}F^{ab}\,. \tag{7.4}$$

This is placed in the AdS_4 black hole background

$$ds^2 = \frac{\ell^2}{z^2}\left(-f(z)dt^2 + \frac{dz^2}{f(z)} + d\vec{x}\cdot d\vec{x}\right) \tag{7.5}$$

with $f = 1 - \left(\frac{z}{z_h}\right)^3$. There is a planar horizon at coordinate value $z = z_h$, while the metric asymptotes to Poincaré AdS. For this calculation we take the bottom-up perspective. However, a concrete top-down approach could be built using M2 branes, as has indeed been the perspective in the original paper [2]. Firstly we fix the axial gauge $A_z = 0$, which is quite convenient for holographic calculations. However, it leaves the residual gauge freedom $A_a \to A_a + \partial_a\Lambda(x^\mu)$, where Λ does not depend on z. This is like a gauge transformation in the boundary theory, and in order to fix it, one must impose a further gauge constraint at a single constant z surface. It is often convenient to use the horizon to do this, but the boundary, or indeed any other constant z surface would do just as well.

Let us work in Fourier space:

$$A_\mu(z, x^\mu) = \int \frac{d^3q}{(2\pi)^3}e^{-i\omega t+iqy}A_\mu(\omega, q), \tag{7.6}$$

where we have oriented $\vec{q} = q\hat{e}_y$ without loss of generality, a reflection of the $SO(2)$ symmetry of the background.

The equations of motion, i.e. the covariant Maxwell equations in the black-hole background, read

$$\mathfrak{w}A_t' + \mathfrak{q}fA_y' = 0 \tag{7.7}$$

$$A_t'' - \frac{1}{f}\left(\mathfrak{q}^2A_t + \mathfrak{w}\mathfrak{q}A_y\right) = 0 \tag{7.8}$$

$$A_x'' + \frac{f'}{f}A_x' - \frac{1}{f}\left(\mathfrak{q}^2 - \frac{\mathfrak{w}^2}{f}\right)A_x = 0 \tag{7.9}$$

$$A_y'' + \frac{f'}{f}A_y' + \frac{\mathfrak{w}^2}{f^2}A_y + \frac{\mathfrak{w}\mathfrak{q}}{f^2}A_t = 0. \tag{7.10}$$

Comments

1. The first equation, (7.7), is a constraint expressing the current Ward identity

$$\partial_\mu \langle J^\mu \rangle = 0\,.$$

This is a general feature: bulk local symmetries – in this case U(1) – lead to boundary Ward identities expressing the conservation of the dual current. Of course if we add sources, the Ward identity will become the continuity equation.

2. For convenience we have gone to dimensionless variables

$$(\omega, q) \to (\mathfrak{w}, \mathfrak{q}) =: z_h(\omega, q)$$

corresponding to the rescaled radial coordinate $z \to \frac{z}{z_h}$ Since $T \sim 1/z_h$ this is saying that we measure all energies in units of temperature from now on.

3. The equation for A_x decoupled from A_y and A_t. In fact, this is a consequence of symmetry: A_x is the transverse (to the momentum vector) component of the gauge field, and so by parity it cannot mix with the remaining ones. A similar analysis will turn out to be very helpful for the later case of the shear viscosity where we will have to deal in addition with the tensor modes coming from the metric perturbation. Here it means we can distinguish two sectors, the longitudinal A_y, A_t sector, and the transverse A_x sector. These correspond to different physical processes in the boundary theory, as we will see presently.

We first solve the A_y, A_t sector. First, solving (7.8) for A_y gives

$$A_y = \frac{f}{\mathfrak{q}\mathfrak{w}} A_t'' - \frac{\mathfrak{q}}{\mathfrak{w}} A_t\,. \tag{7.11}$$

Now, substituting this into (7.10), we get a single third-order differential equation,

$$A_t''' + \frac{f'}{f} A_t'' + \frac{1}{f}\left(\frac{\mathfrak{w}^2}{f} - \mathfrak{q}^2\right) A_t' = 0, \tag{7.12}$$

which is a second order ODE for $\phi = A_t'$ with no analytically known solution. However, we know that the Kubo formula for the transport coefficient only needs $\mathfrak{w}, \mathfrak{q} \ll 1$, so we can develop an expansion in small $(\mathfrak{w}, \mathfrak{q})$. Furthermore we are interested in the retarded correlation function, so we look for the ingoing mode at the horizon

$$\phi \sim (1-z)^{-\frac{i\mathfrak{w}}{3}} F(z), \tag{7.13}$$

then $F(z)$ can be developed as as series in small frequency and momentum

$$F(z) = F_{(0)} + \mathfrak{w} F_{(1,0)} + \mathfrak{q}^2 F_{(0,2)} + \mathfrak{w}^2 F_{(2,0)} + \cdots \tag{7.14}$$

A somewhat tedious, but straightforward, calculation[1] gives the first few terms in this expansion

$$\begin{aligned}
F_{(0)} &= C \\
F_{(1,0)} &= -\frac{C}{6}\left[2\sqrt{3}\tan^{-1}\left(\frac{1+2z}{\sqrt{3}}\right) + \log\left(\frac{1-z^3}{1-z}\right)\right] \\
F_{(0,2)} &= -\frac{2C}{\sqrt{3}}\tan^{-1}\left(\frac{1+2z}{\sqrt{3}}\right)
\end{aligned}$$

for an arbitrary constant C, which we will determine shortly. In order to uniquely determine these we had to require: 1) regularity at the horizon order by order, and 2) that only the leading order contains the homogeneous piece. We can now use (7.11) to determine the constant C in terms of the boundary data of the longitudinal vector modes,

$$C = \frac{\mathfrak{q}^2 A_t^{(0)} + \mathfrak{w}\mathfrak{q} A_y^{(0)}}{i\mathfrak{w} - \mathfrak{q}^2}, \tag{7.15}$$

which is gauge invariant under residual gauge transformations, which is easy to see as the denominator is the Fourier transform of the field strength component

[1] Seek the help of Mathematica, Maple, Dr Valdez.

$F_{yt} = (\partial_y A_t - \partial_t A_y)$. We thus have

$$A_t = A_t^{(0)} + z\,C + O(z^2) \tag{7.16}$$
$$A_y = A_y^{(0)} - z\frac{\mathfrak{w}}{\mathfrak{q}}C + O(z^2) \tag{7.17}$$

Note:

1. $A_{t,y}^{(0)}$ are the boundary data for the fields in the longitudinal sector, that is the equivalent of $\phi_{(0)}$ in previous sections. They have the interpretation as sources for the longitudinal components of the current in the boundary.

2. The terms of order $O(z)$ are *not* determined by near-boundary analysis (the analog of the coefficient $\phi_{(2\Delta-d)}$ in our previous analysis for a scalar operator of dimension $\Delta = 2$, $d = 3$). However, we have been able to determine them (7.15) by finding full bulk solution, linking the solution with ingoing boundary conditions in the infrared to the UV data.

We can now read off the components of the retarded correlation function in the longitudinal sector, namely

$$G^R_{tt}(\mathfrak{w},\mathfrak{q}) = \mathcal{N}\frac{\mathfrak{q}^2}{i\mathfrak{w} - D\mathfrak{q}^2} \tag{7.18}$$
$$G^R_{yy}(\mathfrak{w},\mathfrak{q}) = \mathcal{N}\frac{\mathfrak{w}^2}{i\mathfrak{w} - D\mathfrak{q}^2} \tag{7.19}$$
$$G^R_{ty}(\mathfrak{w},\mathfrak{q}) = \mathcal{N}\frac{\mathfrak{q}\mathfrak{w}}{i\mathfrak{w} - D\mathfrak{q}^2}\,, \tag{7.20}$$

with $\mathcal{N} = \frac{3}{4e^2(2\pi)^3}$. Noteworthy is the location of the pole, as a function of frequency and momentum

$$i\mathfrak{w} = D\mathfrak{q}^2\,, \tag{7.21}$$

where, reinstating units, we find $D = z_h = \frac{3}{4\pi T}$. A relation between frequency and momentum of a pole is known as a dispersion relation, and the form (7.21) is referred to a diffusive pole. This is intuitive, as it is basically the diffusion equation

$$\partial_t n = D\nabla^2 n \tag{7.22}$$

in momentum space. Here n is the density of the diffusing quantity, be it charge or the concentration of dye dropped in a container of water. The physics of the longitudinal sector is thus that of charge diffusion, and using our Lorentzian holographic techniques, we have determined the charge diffusion constant of a strongly coupled $2+1$ quantum field theory. Performing this calculation using conventional techniques (essentially perturbation theory) would be no small feat, but here it was a relatively easy calculation illustrating the power of holography.

In order to find the actual electric conductivity we should solve the equation for A_x, the transverse component. Actually this is now almost trivial, as we have done all the hard work already: the resulting differential equations are of exactly the same form as the ones we solve above. We start again by peeling off the ingoing behavior at the horizon, and posing

$$A_x = (1-z)^{-\frac{i\mathfrak{w}}{3}} F(z), \tag{7.23}$$

where

$$F(z) = F_{(0)} + \mathfrak{w} F_{(1,0)} + \mathfrak{q}^2 F_{(0,2)} + \cdots$$

with coefficient functions $F_{(i,j)}$ exactly as above. Expanding $A_x(z, \mathfrak{w}, \mathfrak{q})$ near $z = 0$, we find

$$F(z) = \frac{C}{18}(18 + i\sqrt{3}\pi\mathfrak{w} - 2\sqrt{3}\mathfrak{q}^2\pi) + C\left(\frac{2i\mathfrak{w}}{3} - \mathfrak{q}^2\right) z + \cdots \tag{7.24}$$

Proceeding as usual to extract the correlator resuls in

$$G^R_{xx}(\mathfrak{w}, \mathfrak{q}) = -\mathcal{N}(i\mathfrak{w} - D\mathfrak{q}^2), \tag{7.25}$$

From the Kubo formula we learn that the coefficient of $\mathfrak{w}$ gives the conductivity, in the present case

$$\sigma^{(2+1)}_{ij} = \delta_{ij}\mathcal{N} \tag{7.26}$$

Comments

1. If we match parameters to the field theory via the top-down M2 brane picture, we find $\mathcal{N} \sim \frac{1}{\ell^2} \sim N^{3/2}$ for $M2$ brane theory. This is an echo of the famous $N^{3/2}$ scaling of the free energy of the M2 brane theory.

2. In summary, our results for charge transport in a strongly coupled $2+1$ dimensional quantum field theory are

$$\begin{aligned} \sigma_{ij} &= \mathcal{N}\delta_{ij} \\ D &= \frac{3}{2\pi T} \end{aligned}$$

We now proceed to the momentum sector, which involves a calculation very similar to the above, but slightly more technically complicated due to the involvement of tensor fluctuations associated with the metric. The general procedure is the same, that is we identify sectors of linear perturbations which we then solve in a low frequency low momentum expansion with ingoing boundary conditions at the horizon. Taking the appropriate limits we extract the transport coefficients.

7.1.2 Momentum Diffusion & Viscosity

As we saw in Chapter 5 , the shear viscosity quantifies the response of a system to a velocity gradient. This results, at the microscopic level, from transport of momentum along the gradient. Thus we have

$$\text{momentum transport} \longleftrightarrow \text{viscosity}$$

The momentum current in a quantum field theory is given by the spatial component T_{0i} of the stress tensor, so we need to consider the stress energy 2-point function of the dual quantum field theory. This is encoded holographically via the bulk metric and its fluctuations

$$T_{\mu\nu}(t,\vec{x}) \quad \leftrightsquigarrow \quad \text{metric mode } g_{ab}(z,x^{\mu})$$

$$G^{R}_{T_{\mu\nu},T_{\rho\sigma}}(t,\vec{x}) \quad \leftrightsquigarrow \quad \text{ingoing mode of } \delta g_{ab} := h_{ab} \tag{7.27}$$

where we choose ingoing boundary conditions because, according to the Kubo formula, we need to determine the retarded correlation function. Because the metric fluctuation is a rank two tensor, we will do well to organise our calculation, using the fact that h_{ab} decomposes into sectors according to the symmetry of the background metric. Again we make the gauge choice $h_{za} = 0$, and work in Fourier space

$$h_{\mu\nu}(z,x^{\mu}) = \int \frac{d^3q}{(2\pi)^3} e^{-i\omega t + iqy} h_{\mu\nu}(\omega,q), \tag{7.28}$$

where the momentum has been oriented as $\vec{q} = q\hat{e}_y$. The radial gauge again does not fully fix the freedom and there will be residual diffeomorphisms. Since the viscosity is determined by the specific components, $\langle T_{xy}, T_{xy}\rangle$, of the stress-stress two-point function, which in turn is dual to the h_{xy} fluctuation, we should keep all modes that mix with h_{xy}. We can use parity along the transverse direction ($x \to -x$) to classify these sectors. The components behave in the following way under parity

$$\begin{aligned} h_{xy} &: \text{ odd} \\ h_{xx} &: \text{ even} \\ h_{yy} &: \text{ even} \\ h_{tx} &: \text{ odd,} \end{aligned}$$

implying that only h_{tx} can mix with h_{xy}. It is convenient to define

$$h_y^x := e^{-i\omega t+iqy}H_y(z)$$
$$h_t^x := e^{-i\omega t+iqy}H_t(z)\,.$$

With these defintions, the linearized Einstein equations give

$$\begin{aligned}
H_t'' - \frac{2}{z}H_t' - \frac{1}{f}(\omega q H_y + q^2 H_t) &= 0 && (7.29)\\
H_y'' + \frac{f-3}{zf}H_y' + \frac{1}{f^2}(\omega^2 H_y + q\omega H_t) &= 0 && (7.30)\\
\omega H_t' + fqH_y' &= 0\,, && (7.31)
\end{aligned}$$

which are very similar in structure to what we saw in the case of charge transport. Proceeding as in that case, one finds, in terms of the dimensionless variables $\mathfrak{w}, \mathfrak{q}$

$$H_t''' + \frac{f-3}{xf}H_t'' + \left[\frac{1}{f^2}(\mathfrak{w}^2 - f\mathfrak{q}^2) - \frac{4}{z^2} + \frac{6}{z^2 f}\right]H_t' = 0. \tag{7.32}$$

Taking $H_t' = \Phi$ and stripping off the ingoing behavior at the horizon

$$\Phi = (1-z)^{-\frac{i\omega}{3}}F(z)$$

the expansion for $\mathfrak{w}, \mathfrak{q} \ll 1$ becomes

$$F(z) = C\left\{z^2 + i\omega\left[z(z-1) + \frac{z^2}{6}f_1(z) - \frac{z^2}{\sqrt{3}}f_2(z)\right] + \frac{q^2}{3}z(1-z)\right\}, \tag{7.33}$$

where

$$\begin{aligned}
f_1(z) &= \log\left(\frac{1}{3}(1+z+z^2)\right)\\
f_2(z) &= \tan^{-1}\left(\frac{1+z^2}{\sqrt{3}}\right)\,. && (7.34)
\end{aligned}$$

Exactly as in the conductivity case, we consider the constraint equation (7.29) near $z = 0$ to determine the constant

$$C = \frac{\mathfrak{q}\mathfrak{w}H_y^{(0)} + \mathfrak{q}^2 H_t^{(0)}}{i\mathfrak{w} - \frac{1}{3}\mathfrak{q}^2}. \tag{7.35}$$

This time let us read off the two-point functions from the on-shell action, which looks like

$$S_{\text{on-shell}} = \frac{2^{5/2}\pi^2}{3^4} N^{3/2} T^3 \int dz\, d^3x \frac{1}{z^2} \left((H_t')^2 - f(H_x')^2 + \cdots \right) \tag{7.36}$$

Thus we have the correlators

$$G_{ty,ty} = \mathcal{N} \frac{\mathfrak{q}^2}{i\mathfrak{w} - D\mathfrak{q}^2} \tag{7.37}$$

$$G_{ty,xy} = -\mathcal{N} \frac{\mathfrak{w}\mathfrak{q}}{i\mathfrak{w} - D\mathfrak{q}^2} \tag{7.38}$$

$$G_{xy,xy} = \mathcal{N} \frac{\mathfrak{w}^2}{i\mathfrak{w} - D\mathfrak{q}^2} \tag{7.39}$$

with

$$\mathcal{N} = \frac{2^{2/3}\pi N^{3/2} T^2}{3^3}, \qquad D = \frac{1}{4\pi T}. \tag{7.40}$$

The Kubo formula then gives the shear viscosity

$$\eta = \frac{2^{3/2\pi}}{3^3} N^{3/2} T^2. \tag{7.41}$$

We now recall from previous chapters our result for the entropy density of the planar black hole in AdS, which in field-theory units reads

$$s = \frac{9\sqrt{2}\pi^2 N^{3/2}}{27} T^2 \tag{7.42}$$

Hence the ratio of shear viscosity to entropy density comes out to be

$$\frac{\eta}{s} = \frac{2^{2/3}\pi\, 27}{3^3 8\sqrt{2}\pi^2} = \frac{1}{4\pi} \tag{7.43}$$

which is the famous result for $\mathcal{N} = 4$ SYM in $3 + 1$ and evidently also holds in

$2 + 1$ dimensional strongly coupled field theory, e.g. the ABJM theory.

Bibliography

[1] G. Policastro, D. T. Son and A. O. Starinets, *"The Shear Viscosity of Strongly Coupled $\mathcal{N} = 4$ Supersymmetric Yang-Mills Plasma,"* Phys. Rev. Lett. **87** (2001) 081601 doi:10.1103/PhysRevLett.87.081601 [hep-th/0104066].

[2] C. P. Herzog, *"The Hydrodynamics of M Theory,"* JHEP **0212** (2002) 026 doi:10.1088/1126-6708/2002/12/026 [hep-th/0210126].